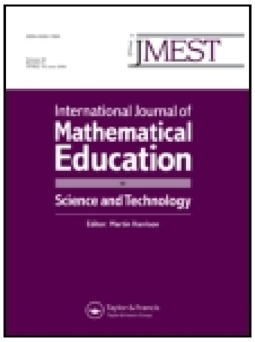

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Classroom notes: Summing sequences having mixed signs

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5. Conclusions

The above birth equations show the pattern of the dynamics of inheritance of sickle-cell anaemia. The *a priori* estimates of the continuous type was obtained in [6]. We are yet to make use of the discrete form and hope this will be done in the near future.

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Summing sequences having mixed signs

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A result is discussed which permits the summing of series whose terms have more complicated sign patterns than simply alternating plus and minus. The Alternating Series Test, commonly taught in beginning calculus courses, is a

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corollary. This result, which is not difficult to prove, widens the series summable by beginning students and paves the way for understanding more advanced questions such as convergence of Fourier series. An elementary exposition is given of Dirichlet's Test for the convergence of a series and an elementary example suitable for a beginning calculus class and a more advanced example involving a Fourier series which is appropriate for an advanced calculus class are provided. Finally, two examples are discussed for which Dirichlet's Test does not apply and a general procedure is given for deciding the convergence or divergence of these and similar examples.

1. Introduction

The subject of infinite series is deep and to many is the hardest part of beginning calculus. One reason for this is that the beginning student has to grasp the notion that an infinite series is really a limit, and many calculations require having a repertoire of known convergent and divergent series in order to apply the comparison test or limit comparison test. The notions are quite subtle and are not fully presented to many students until an advanced calculus course.

The introduction to summing an infinite sequence of real numbers

$$c_1, c_2, \ldots, c_n, \ldots$$

given in most calculus texts ([1, 3] are representative) runs roughly in the following sequence:

(i) the definition of an infinite series as a limit of partial sums

$$\sum_{n=1}^{\infty} c_n = \lim_{n\to\infty} (c_1 + c_2 + \ldots + c_n)$$

(ii) a discussion of geometric series

$$\sum_{n=1}^{\infty} aR^{n-1}$$

(iii) a discussion of the result that if $\sum_{n=1}^{\infty} c_n$ is convergent, then $\lim_{n\to\infty} c_n = 0$;
(iv) the arithmetic of convergent series;
(v) series of positive terms, the Integral Test, p-Series, the Comparison Test and perhaps the Limit Comparison Test;
(vi) the Alternating Series Test;
(vii) absolute convergence versus conditional convergence, the Ratio Test and perhaps the Root Test.

All of this is done so that the behaviour of Taylor and Maclaurin series can be investigated and explained. There are many other applications as well, most notably decimal (and other bases) representations of real numbers, finding areas, and defining important functions such as the Bessel functions of mathematical physics. Of course, a good grasp of these fundamentals is essential in understanding the behaviour of Fourier series and the difference between pointwise convergence and uniform convergence (the latter notions usually are not discussed for Taylor's series at the beginning level). More sophisticated tests such as the Comparison Ratio Test, D'Alembert's Ratio Test, Kummer's Test, Raabe's Test, and Gauss' Test developed to handle series for which the Ratio Test fails are never mentioned.

After all this development, the beginning student still might carry away the impression that infinite series are composed of either all positive terms, or failing that, are alternating series; series with other sign patterns are simply not discussed at all and hence must be unimportant. For example, students are able to deduce the (conditional) convergence of the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

from the Alternating Series Test, but have no training to help them decide about the convergence of the series

$$1+\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\dots \tag{1.1}$$

which is the series formed with the terms $1/n$ and the sign pattern

$$+++---+++---\dots$$

In this article, we deduce a remarkably simple result that permits us to decide about the convergence of many series with non-alternating non-constant sign patterns which will include the Alternating Series Test as an immediate corollary. This result has a straightforward proof, which we forego in favour of proving a more general result called Dirichlet's Test. This latter result is fundamental in the study of Fourier series and perhaps needs more prominent mention than is usual and by proving it we keep this article self-contained.

2. Dirichlet's Test

Let us begin by mentioning a Fourier series that arises as the solution to a one-dimensional heat equation[1] for $-1 \leq x \leq 1$.

$$\cos(\pi x/2)-\frac{1}{3}\cos(3\pi x/2)+\frac{1}{5}\cos(5\pi x/2)-\frac{1}{7}\cos(7\pi x/2)+\dots \tag{2.1}$$

This series, at first glance, appears to be alternating and is convergent at $x=0$ since the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$$

is well known to be (conditionally) convergent to arctan(1). We see that the sign pattern is

$$+-+-+\dots$$

for $x=0$, reinforcing the impression that the Fourier series (2.1) is alternating. But if $x=0.3$, then the sign pattern is quite different

[1]For more details on this and a treatment of Fourier analysis at the undergraduate level from a historical viewpoint, we refer the reader to the interesting text by David Bressoud, *A Radical Approach to Real Analysis*, 1994, published by The Mathematical Association of America [2].

$$\begin{array}{l} + - - + - - + - - + \\ - + + - + + - + + - \\ + - - + - - + - - + \\ - + + \ldots \end{array}$$

and thus the question of convergence of this series for $x = 0.3$ is not covered by any results from beginning calculus.

The key insight into studying the convergence of this Fourier series and the series (1.1) is to consider them to be of the form

$$\sum_{n=1}^{\infty} a_n b_n \tag{2.2}$$

where the sequence of b_ns is positive valued, monotonically decreasing, and has limit 0, and the sequence of a_ns, while less predictable, is well behaved in that the partial sums remain bounded. It is useful to single out an important observation for such series.

Abel's Lemma (1826). Consider a series of the form given in (2.2) where

$$b_1 \geq b_2 \geq \ldots \geq 0$$

and denote the partial sums for the a_ns by

$$S_n = \sum_{k=1}^{n} a_k$$

Suppose that there is a positive number M for which

$$|S_n| \leq M$$

for all values of n. Then the partial sums of the series (2.2) are bounded by the value Mb_1; that is,

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq M b_1$$

for all values of n.

Proof. Since $a_n = S_n - S_{n-1}$, we can rewrite the partial sum for the series (2.2) as follows

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k &= S_1 b_1 + (S_2 - S_1) b_2 + \ldots + (S_n - S_{n-1}) b_n \\ &= (S_1 b_1 + S_2 b_2 + \ldots + S_n b_n) - (S_1 b_2 + S_2 b_3 + \ldots + S_{n-1} b_n) \\ &= \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}) + S_n b_n \end{aligned}$$

Since $b_k - b_{k+1} \geq 0$, it now follows easily from repeated use of the triangle inequality that

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k b_k\right| &\leq \sum_{k=1}^{n-1} |S_k|(b_k - b_{k+1}) + |S_n| b_n \\ &\leq \sum_{k=1}^{n-1} M(b_k - b_{k+1}) + M b_n \\ &= M(b_1 - b_2 + b_2 - b_3 + \ldots + b_{n-1} - b_n + b_n) \\ &= M b_1\end{aligned}$$

So far the treatment is very elementary, but to step beyond Abel's Lemma and obtain a convergence test, we must invoke the Cauchy criterion which permits us to decide precisely when a series is convergent.

Cauchy Criterion (1821). Let $c_1 + c_2 + \ldots + c_n + \ldots$ be an infinite series with partial sums

$$S_n = \sum_{k=1}^{n} c_k$$

This series is convergent if and only if for every $\varepsilon > 0$, there is an N so that for all pairs n and m with $n > m > N$, we have

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} c_k\right| < \varepsilon$$

Under the additional assumption (not assumed in Abel's Lemma) that the b_ns converge to 0, we can prove convergence of the series (2.2).

Dirichlet's Test (1829). Consider a series of the form given in (2.2) where

$$b_1 \geq b_2 \geq \ldots \geq 0$$

and

$$\lim_{n\to\infty} b_n = 0$$

Denote the partial sums for the a_ns by

$$S_n = \sum_{k=1}^{n} a_k$$

Suppose that there is a positive number M for which

$$|S_n| \leq M$$

for all values of n. Then the series (2.2) converges.

Proof. The argument is easy and follows from the Cauchy criterion. Suppose $\varepsilon > 0$ is given. By repeating the argument used in Abel's Lemma,

$$\left|\sum_{k=n+1}^{m} a_k b_k\right| \leq M b_{n+1}$$

Since the b_ns converge to 0, there is an N so that for all $k > N$, $b_k < \varepsilon/M$. Thus if $n+1 > N$, we have

$$\left|\sum_{k=n+1}^{m} a_k b_k\right| < M\frac{\varepsilon}{M} = \varepsilon$$

Consequently, by the Cauchy Criterion, the series (2.2) is convergent.

Now let us see how Dirichlet's Test helps with the series (1.1). We let $b_n = 1/n$, and a_n equal $+1$ or -1 according to the appropriate sign pattern position for the nth term. Then it is easy to see that the absolute value of the partial sums of the a_ns is bounded by 3. More precisely, we have the following corollary.

Corollary. Suppose that $\{b_k\}_{k=1}^{\infty}$ is a positive, decreasing sequence which converges to zero. Then, $\sum_{n=1}^{\infty} a_n b_n$ converges where for each i, $a_i = \pm 1$, if

(1) $\sum_{n=1}^{\infty} b_n$ is convergent (and thus the original series is absolutely convergent), or
(2) $\sum_{n=1}^{\infty} b_n$ is not convergent, but $\{\sum_{n=1}^{N} a_n\}_{N=1}^{\infty}$ is a bounded sequence.

Thus any series having a well-defined sign pattern for which the partial sums of the $+1$s and -1s are bounded is convergent. This is an easy test which beginning students could easily apply. While the notion of the Cauchy Criterion may be too advanced for the beginning student, our corollary certainly is suitable as a statement of fact (as many of the theorems in beginning calculus are often presented).

Corollary. Alternating Series Test. If $c_1, c_2, \ldots, c_n, \ldots$ is a sequence of terms alternating in sign and decreasing in absolute value, then the series

$$\sum_{n=1}^{\infty} c_n$$

is convergent if and only if

$$\lim_{n\to\infty} c_n = 0$$

Finally, Dirichlet's Test also applies to the Fourier series (2.1). This series is defined for all values of x, not just those in the unit interval. For example, at any odd integer the series sums to 0. If x is not an odd integer, then we let $a_n = (-1)^n \cos((2n-1)\pi x/2)$ and $b_n = 1/(2n-1)$. To see that the series is convergent (this is far from obvious, the great mathematician Joseph Louis Lagrange thought that the series diverged), all that we need to observe is that the partial sums

$$\sum_{k=1}^{n} (-1)^k \cos((2k-1)\pi x/2)$$

stay bounded. A non-elementary trigonometric identity reduces the difficulty. It follows that

$$\sum_{k=1}^{n}(-1)^k \cos((2k-1)\pi x/2) = \frac{1-(-1)^n \cos(\pi n x)}{2\cos(\pi x/2)}$$

and thus

$$\left|\sum_{k=1}^{n}(-1)^k \cos((2k-1)\pi x/2)\right| \leq |\sec(\pi x/2)|$$

for all x not an odd integer. The series (2.1) converges pointwise for all values of x.

3. Two examples

The study of series having mixed signs is far from as simple as one might think from the above discussion for there is no reason to expect the partial sums of the +1s and −1s to be bounded. The point of the next two examples is to show that if the partial sums of the +1s and −1s arising from a sign pattern is unbounded, then the convergence or divergence of the series is undetermined; the same sign pattern can occur in both cases. Consider the series

$$1+\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}-\dots \tag{3.1}$$

Here the sign pattern is

$$+++-----+++++++---------+\dots$$

We have three +s followed by five −s, followed by seven +s, followed by nine −s, and so on. Clearly the partial sums of +1s and −1s for this series are unbounded.

Let us call the groups of consecutive $1/n$s all having the same sign a *cohort*. The first cohort is $\{1, 1/2, 1/3\}$ and the second is $\{1/4, 1/5, 1/6, 1/7, 1/8\}$, etc. If we form a new series by summing the terms of each cohort, then we obtain an alternating series

$$\frac{11}{6}-\frac{743}{840}+\frac{21635}{36036}-\frac{188522063}{411863760}+\dots \tag{3.2}$$

whose partial sums form a subsequence of the partial sums of series (3.1). The nth term of this series is

$$(-1)^{n+1}\left(\frac{1}{n^2}+\dots+\frac{1}{n^2+2n}\right)$$

which, in absolute value, is bounded above by $(2n+1)/n^2$. Thus it follows that the limit of the sequence of terms is zero and this alternating series is convergent.

To see that the series (3.1) is convergent, let us denote the partial sums of the series (3.1) by S_n and the partial sums of the series (3.2) by S_n^*. Observe that a partial sum S_n for (3.1) consists of a partial sum $S_{n_1}^*$ of (3.2) plus or minus a finite sum of consecutive terms of the form

$$\frac{1}{k}+\frac{1}{k+1}+\dots+\frac{1}{k+p} \tag{3.3}$$

The difference between two partial sums for (3.1) is the difference between two partial sums for (3.2) and two finite sequences of the form (3.3). Thus

$$|S_n - S_m| \leq \left|S^*_{n_1} - S^*_{m_1}\right| + \left|\frac{1}{k_n} + \frac{1}{k_n+1} + \ldots + \frac{1}{k_n+p_n}\right| + \left|\frac{1}{k_m} + \frac{1}{k_m+1} + \ldots + \frac{1}{k_m+p_m}\right| \tag{3.4}$$

Using the fact that $\lim_{n\to\infty}(1/n) = 0$ and the Cauchy criterion for the convergent series (3.2), given an $\varepsilon > 0$, one can find an N so that for all $n, m > N$, each term of the right-hand side of the inequality (3.4) is less than $\varepsilon/3$. The details of this are left to the reader. This means the series (3.1) converges by the Cauchy criterion.

Next consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} - \ldots \tag{3.5}$$

This series has the same sign pattern as the previous example. Again we form an alternating series by summing the terms of the cohorts

$$\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}}\right) + \ldots \tag{3.6}$$

The partial sums of this series forms a subsequence of the partial sums for the series (3.5). The nth term of this alternating series:

$$\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+2n}}$$

is bounded below by

$$\frac{2n+1}{\sqrt{n^2+2n}}$$

and this term tends to 2 as n tends to infinity. The alternating series is divergent and its sequence of partial sums fails to converge. This means that the sequence of partial sums for (3.5) fails to converge as well and consequently (3.5) is divergent.

The strategy of proof indicated above can be applied to other examples with similar sign patterns. If, by summing the cohorts, one obtains a convergent series, then the original series is convergent as well; if one obtains a divergent series, then the original series is divergent.

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