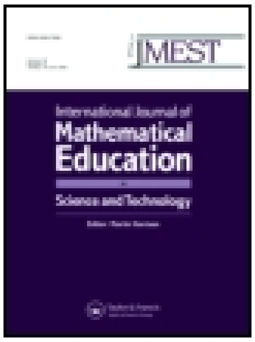

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International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: http://www.tandfonline.com/loi/tmes20

Classroom note: A simple non-trivial student model of a bouncing soccer ball

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Proof. Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ is finite, $g(s,y)$ is an integrable function of s for each y. Therefore, by Lebesgue's Theorem [see Folland ([6], p. 93)] on differentiation, $\partial F/\partial x$ exists a.e., and

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} &= \frac{\partial}{\partial x}\int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{-\infty}^{y} f(x,t)\,\mathrm{d}t \quad \text{a.e.}\end{aligned}$$

Moreover, since for almost every x, $f(x,t)$ is an integrable function of t, another application of Lebesgue's Theorem yields that $\partial^2 F/\partial y \partial x$ exists a.e., and

$$\frac{\partial^2 F(x,y)}{\partial y \partial x} = f(x,y) \quad \text{a.e.}$$

Finally, by reversing the order of integration and following the same procedure, we find that $\partial F/\partial y$ and $\partial^2 F/\partial x \partial y$ exist a.e., and

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y) \quad \text{a.e.}$$

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A simple non-trivial student model of a bouncing soccer ball

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(*Received 30 August 2002*)

When students have made some progress in studying ordinary differential equations (ODEs) during a semester, small groups are given a soccer ball and asked to determine the simplest model describing one bounce of the ball from shoulder height. To start with, they play with the ball. However, after a few weeks, they begin to ask penetrating questions and they are led to a simple model that has a non-trivial twist to the solution.

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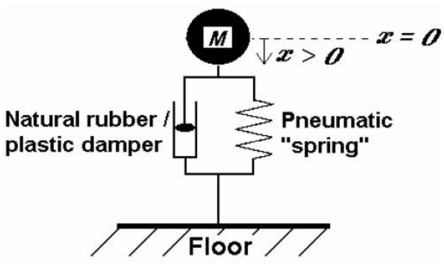

Figure 1. The spring–damper pair.

1. Introduction

Typically, our students have two semesters of calculus before they register for an introductory ODE course. Once the concept of second-order ODEs with constant coefficients has been mastered and a few easy exercises have been done, we introduce the idea of a spring–damper pair, with the accompanying concepts of overdamped, underdamped and critically damped free motion. During this period of conceptual development, we give small groups of about five students a soccer ball. Initially, we do not give them any information on what we expect them to do. All we require is that the balls be brought to the ODE class each and every time. About a week later, when the class next meets, we ask them to model one bounce of the ball dropped vertically from shoulder height.

To start with, they play with the ball. However, after a few weeks, they begin to ask penetrating questions and we lead them to a simple model that has a non-trivial twist to the solution, as described in the next sections.

2. A non-trivial simplistic model

The most accurate view of a soccer ball would be to see it as a distributed parameter problem described by a PDE (partial differential equation), which is usually solved using finite-element techniques. The first step in the process of simplifying the model is to reduce it to a (large) set of coupled ODEs. This, however, is still a computationally complex problem to solve. A further simplification would be to model the soccer ball by a single point mass concentrated at the centre of the ball, surrounded by a large number of spring–damper pairs. The next step is to restrict attention to vertical motion of the ball only, which would enable us to neglect all the spring–damper pairs except for the two pairs directly above and below the point mass. By replacing these two spring–damper pairs by a single spring–damper configuration, the simplest possible model, a second-order ODE, is obtained. The parameters of this model are the mass M, the spring constant K and the internal frictional coefficient B of the ball (see figure 1).

The ODE to consider solving for this simplified model is thus

$$M\ddot{x} + B\dot{x} + Kx = 0 \tag{1}$$

where we have assumed that damping is viscous, that is, it is directly proportional to the velocity of the mass through the medium. Measuring the spring constant K is easy. Simply place the ball on top of a bathroom scale and push downwards. Hold the position at a reading of, say, 20 kg on the scale. Measure, as accurately as a ruler allows, the distance through which a (plastic) soccer ball is compressed.

We found this distance to be approximately $\Delta x = 6\,\text{cm}$. The force used for this compression was thus $F = 20g$ N and so, from Hooke's law $F = K\Delta x$

$$K = \frac{20g}{\Delta x} = \frac{20 \times 9.8}{0.06} \approx 3.3 \times 10^3\,\text{N}\,\text{m}^{-1} \tag{2}$$

If the rubber/plastic damper of the ball were like a shock absorber on a motor car where we assume that the damping force is viscous (i.e. is it is directly proportional to the velocity of the mass through the medium) we could clamp it vertically in a vice, stand on a bathroom scale and pull it out (upwards), measuring our apparent increase in mass Δm as well as the velocity v at which it is pulled out (see [1]). Hence we would have

$$B = \frac{\Delta mg}{v} \tag{3}$$

However, we cannot measure the viscous damping coefficient B directly for a soccer ball because we have no 'shock absorber' to pull on! How do we set about determining B by means of some simple experiment?

3. The coefficient of viscous damping

As suggested in [2], consider the following facts. If one drops the ball through a height $h_1 = 1.5\,\text{m}$, it will strike the floor at some velocity v_1 which we can determine from the well-known equation

$$v_1^2 = 2gh_1 \tag{4}$$

because air friction is negligible for this experiment. It is observed that the ball bounces back up to a height $h_2 = 0.57\,\text{m}$ and we can determine the rebound velocity v_2 from

$$v_2^2 = 2gh_2 \tag{5}$$

Consider that during the time interval between the ball striking the floor and leaving it on the rebound, the behaviour of the ball can be modelled by the damped spring IVP (initial value problem):

$$\left.\begin{aligned} M\ddot{x} + B\dot{x} + Kx = 0 \\ x(0) = 0, \quad \dot{x}(0) = v_1 \end{aligned}\right\} \tag{6}$$

where, as usual, $\dot{x} = (\mathrm{d}x//\mathrm{d}t)$ and $\ddot{x} = (d^2x//\mathrm{d}t^2)$. For convenience, let us rewrite relations (6) as

$$\left.\begin{aligned} \ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0 \\ x(0) = 0, \quad \dot{x}(0) = v_1 \end{aligned}\right\} \tag{7}$$

where

$$2\lambda = \frac{B}{M} \tag{8}$$

and

$$\omega^2 = \frac{K}{M} \tag{9}$$

The characteristic equation of (7) has roots

$$-\lambda \pm \sqrt{\lambda^2 - \omega^2} \tag{10}$$

3.1. *The overdamped case*

Suppose we have **overdamping** ($\lambda^2 > \omega^2$) (see [3] or [4]). Hence we have two negative real roots

$$a = -\lambda + \sqrt{\lambda^2 - \omega^2} < 0 \tag{11}$$

and

$$b = -\lambda - \sqrt{\lambda^2 - \omega^2} < 0 \tag{12}$$

with

$$a - b = 2\sqrt{\lambda^2 - \omega^2} > 0 \tag{13}$$

This yields a general solution to (7) of the form

$$x = \alpha e^{at} + \beta e^{bt} \tag{14}$$

The initial condition $x(0) = 0$ yields:

$$x = \alpha\left(e^{at} - e^{bt}\right) \tag{15}$$

There is only one root of $x(t)$, namely $t = 0$. This follows from equating (15) to zero and then consulting (13). Furthermore,

$$\dot{x} = \alpha\left(ae^{at} - be^{bt}\right) \tag{16}$$

and so the initial condition $\dot{x}(0) = v_1$ yields

$$\dot{x} = \frac{v_1}{2\sqrt{\lambda^2 - \omega^2}}\left(ae^{at} - be^{bt}\right) \tag{17}$$

As usual, in order to determine the maximum displacement (maximum compression) of the ball, we equate (17) to zero. This yields only one solution $t = t_1$ given by

$$t_1 = \frac{1}{2\sqrt{\lambda^2 - \omega^2}} \ln \frac{b}{a} > 0 \tag{18}$$

From (17) we obtain

$$\ddot{x} = \frac{v_1}{2\sqrt{\lambda^2 - \omega^2}}\left(a^2 e^{at} - b^2 e^{bt}\right) \tag{19}$$

Even though we must have $\ddot{x}(t_1) < 0$, if one substitutes (18) into (19) and one simplifies the expression, this fact is not obvious. Hence the second derivative test is not easy to perform, and is, as in most cases of a physical occurrence, not necessary to perform.

By equating (19) to zero we deduce that there is only one inflection point which occurs at time

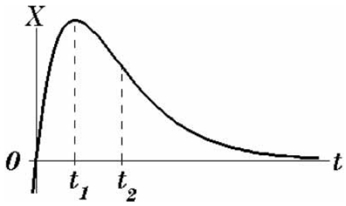

Figure 2. Maximum and inflection points.

$$t_2 = \frac{1}{2\sqrt{\lambda^2 - \omega^2}} \ln \frac{b^2}{a^2} > 0 \tag{20}$$

Consequently,

$$\lim_{t\to\infty} x = 0 \tag{21}$$

and

$$\lim_{t\to\infty} \dot{x} = 0 \tag{22}$$

A graph of this situation is given in figure 2.

Hence $v_2 = 0$ and the ball will not rebound (bounce). Overdamping is therfore not possible.

3.2. *The critically damped case*

Student problem. Assume that we have **critical damping** ($\lambda^2 = \omega^2$). Show that the solution to the IVP (7) is of the form

$$x = e^{-\lambda t}(\alpha + \beta t) \tag{23}$$

with only one root at $t = 0$, a maximum value at $t_1 = 1//\lambda$ and an inflection point at $t_2 = 2//\lambda$. Note that it is easy to perform the second derivative test here, even though physical reality does not necessitate it. Hence the limits (21) and (22) are valid, and so a graph of the situation is similar to figure 2. Conclude that the ball does not bounce, and critical damping is thus not possible.

3.3. *The underdamped case*

Consequently, the only possibility is that there is **underdamping** ($\omega^2 > \lambda^2$) and so equation (7) has a general solution of the form

$$x = e^{-\lambda t}(\alpha \cos \Omega t + \beta \sin \Omega t) \tag{24}$$

where

$$\Omega = \sqrt{\omega^2 - \lambda^2} \tag{25}$$

Applying the first initial condition $x(0) = 0$ to equation (24), we obtain

$$x = \beta e^{-\lambda t} \sin \Omega t \tag{26}$$

Hence

$$\dot{x} = \beta e^{-\lambda t}(\Omega \cos \Omega t - \lambda \sin \Omega t) \tag{27}$$

Applying the second initial condition (7) to equation (27) we obtain

$$x = \frac{v_1}{\Omega} e^{-\lambda t} \sin \Omega t = \frac{\sqrt{2gh_1}}{\Omega} e^{-\lambda t} \sin \Omega t \tag{28}$$

and so

$$\dot{x} = \frac{v_1}{\Omega} e^{-\lambda t} (\Omega \cos \Omega t - \lambda \sin \Omega t) \tag{29}$$

Keeping in mind that the rebound velocity v_2 occurs when $x(t) = 0$ for the second time, it follows from equation (28) that

$$\dot{x}\left(\frac{\pi}{\Omega}\right) = v_2 \tag{30}$$

Hence from equation (29) we obtain

$$v_2 = -v_1 \exp\left(-\frac{\lambda \pi}{\Omega}\right) \tag{31}$$

The ratio

$$R = \left|\frac{v_2}{v_1}\right| \tag{32}$$

is well known in physics and is called the **coefficient of restitution** of the ball, and equation (31) yields

$$R = \exp\left(-\frac{\lambda \pi}{\Omega}\right) \tag{33}$$

From equations (4) and (5) we obtain

$$R = \sqrt{\frac{h_2}{h_1}} \approx 0.62 \tag{34}$$

because, as mentioned above, by dropping the ball through a height $h_1 = 1.5\,\text{m}$, the maximum rebound height off a solid floor is $h_2 = 0.57\,\text{m}$. But equations (33) and (25) yield

$$-\frac{\lambda \pi}{\sqrt{\omega^2 - \lambda^2}} = \ln R \tag{35}$$

Using a kitchen scale, the mass of the plastic soccer ball is found to be

$$M = 0.11\,\text{kg} \tag{36}$$

Hence, keeping equation (9) in mind, from equation (35) we obtain

$$\lambda = -\omega \frac{\ln R}{\sqrt{\pi^2 + \ln^2 R}} \approx 26\,\text{Hz} \tag{37}$$

Consequently, from equation (8), we obtain the illusive coefficient of damping

$$B = 2\lambda M \approx 5.8\,\text{kg}\,\text{s}^{-1} \tag{38}$$

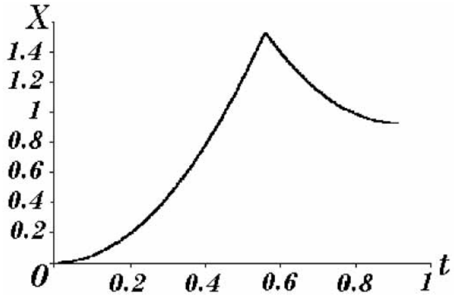

Figure 3. Graph of the model function $X(t)$.

4. The complete model

Because air friction is negligible in this situation, while the ball drops, it obeys the well-known equation

$$h = \frac{1}{2}gt^2 \tag{39}$$

Consider that it takes approximately

$$t_3 = \sqrt{\frac{2h_1}{g}} \approx 0.553\,\mathrm{s} \tag{40}$$

for the ball to fall through height h_1. Similarly, after a further short time

$$t_4 = \frac{\pi}{\Omega} \approx 1.8 \times 10^{-2}\,\mathrm{s} \tag{41}$$

the ball rebounds up through a height h_2 during the time

$$t_5 = \sqrt{\frac{2h_2}{g}} \approx 0.34\,\mathrm{s} \tag{42}$$

Keeping equations (39) and (28) in mind, the function describing the position of the ball below the position from which it is dropped is

$$X(t) = \begin{cases} \frac{1}{2}gt^2 & \text{if } 0 \leqslant t < t_3 \\ h_1 + \frac{\sqrt{2gh_1}}{\Omega} e^{-\lambda(t-t_3)} \sin \Omega(t - t_3) & \text{if } t_3 \leqslant t < t_3 + t_4 \\ h_1 - h_2 + \frac{1}{2}g(t - t_3 - t_4 - t_5)^2 & \text{if } t_3 + t_4 \leqslant t \leqslant t_3 + t_4 + t_5 \end{cases} \tag{43}$$

A graph of $X(t)$ is given in figure 3.

5. Conclusions

Working through this student model will introduce and/or reinforce the following mathematical and/or educational concepts in a natural and topical setting:

(1) a simplified model
   1. of a well-known physical occurrence
   2. with a realistic solution

(2) a non-trivial (non-'plug and chug') problem
(3) spring–damper pairs with the entire spectrum of the overdamped, critically damped and underdamped situations
(4) maximum, minimum and inflection points
(5) parabolic trajectories
(6) horizontal translation of graphs
(7) a realistic, piecewise defined function
(8) small group work

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