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Optical solitons in birefringent fibers by extended trial equation method

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ABSTRACT

This paper studies solitons in optical fibers by the aid of extended trial function method. This powerful mathematical tool is applied to birefringent fibers with Kerr and parabolic laws of nonlinearity. Bright and singular soliton solutions are obtained along with respective constraints that must hold to guarantee the existence of these solitons. As a byproduct, singular periodic and doubly periodic solutions are also recovered from this algorithm.

1. Introduction

Optical solitons is one of the most fascinating areas of research in the field of nonlinear optics. In fact, solitons are treasure-trove for long distance fiber-optic communication along trans-continental and trans-oceanic distances. However, when rubber meets the road, there are several problems one encounters. These are pulse dissipation and attenuation, evolution of ghost pulses, collision-induced frequency and timing jitter and many more. This paper will discuss the consequence of pulse splitting. When a pulse is injected into the optical fiber at the initial end, it tends to split into two orthogonal components. This occurs due to random variation of fiber diameter, twists and bends as well as other unwanted features that naturally cause these polarization. The splitting of pulses lead to differential group delay and this phenomena is called birefringence in optical fibers.

The basic fabric of soliton propagation is the nonlinear Schrödinger's equation (NLSE) [1–20]. For birefringent fibers, the dynamics of these soliton propagation is modeled by vector coupled NLSE where the two components of NLSE represent split pulse components. The coupling between the two components is maintained by cross-phase modulation. There are several perturbation terms that are considered in this paper, although four-wave mixing (FWM) terms are neglected. Two kinds of optical fibers are studied in this paper. These are Kerr law nonlinearity and parabolic law nonlinear medium.

The aim of this paper is to extract exact 1-soliton solution for the coupled NLSE by using a powerful integration tool. It is the extended trial equation method. This algorithm extracts bright and singular soliton solutions in birefringent fibers along

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with respective constraint conditions that guarantee the existence of these solitons. This integration scheme, as a byproduct, also reveals additional solutions such as singular-periodic and doubly periodic solutions that are not studied in nonlinear optics. The details are discussed in the upcoming sections.

2. Kerr law nonlinearity

This section bright and singular soliton solutions in birefringent optical fibers with Kerr law nonlinearity. The dimensionless form of the coupled NLSE with Kerr law nonlinearity is given by [1,2]

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + (c_1|q|^2 + d_1|r|^2)q + i\{\alpha_1 q_x + \beta_1(|q|^2 q)_x + \nu_1(|q|^2)_x q + \theta_1|q|^2 q_x + \gamma_1 q_{xxx}\} = 0, \tag{1}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + (c_2|r|^2 + d_2|q|^2)r + i\{\alpha_2 r_x + \beta_2(|r|^2 r)_x + \nu_2(|r|^2)_x r + \theta_2|r|^2 r_x + \gamma_2 r_{xxx}\} = 0. \tag{2}$$

In (1) and (2), $q(x, t)$ and $r(x, t)$ are complex valued functions that represents the soliton profiles for the two components in birefringent fibers. For $l = 1, 2$, a_l represents group-velocity dispersion (GVD) and b_l is the coefficient of spatio-temporal dispersion (STD) along the two components. These STD terms are necessary to be included, since it induces the model to be well-posed as reported during 2012 [8,9]. However, NLSE with STD was already studied in the past with STD [18]. Next, c_l and d_l represents the self-phase modulation (SPM) and cross-phase modulation (XPM) respectively. From perturbation terms α_l represents the inter-modal dispersion, β_l is the self-steepening term, ν_l and θ_l are nonlinear dispersions and finally γ_l is the third order dispersion that must be included for small GVD.

The extended trial function method method [10–15,17] is implemented. To start off, $q(x, t)$ and $r(x, t)$ are written in the phase-amplitude format as [1,2]

$$q(x, t) = P_1(x, t)e^{i\phi_1(x,t)} = P_1(x, t)e^{i(-\kappa_1 x + \omega_1 t + \sigma_1)}, \tag{3}$$

and

$$r(x, t) = P_2(x, t)e^{i\phi_2(x,t)} = P_2(x, t)e^{i(-\kappa_2 x + \omega_2 t + \sigma_2)}, \tag{4}$$

where P_l for $l = 1, 2$ are the amplitude components of the solitons and $\phi_l(x, t)$ are its phase components, defined as

$$\phi_l(x, t) = -\kappa_l x + \omega_l t + \sigma_l. \tag{5}$$

Here, κ_l are the frequencies of the solitons in each of the two components, ω_l are the wave numbers, while σ_l are the phase constants. Substituting (3) and (4) into (1) and (2) and decomposing into real and imaginary parts lead to

$$(\omega_l + a_l\kappa_l^2 - b_l\kappa_l\omega_l - \alpha_l\kappa_l + \gamma_l\kappa_l^3)P_l - d_l P_l P_{\bar{l}}^2 - (\beta_l\kappa_l + \theta_l\kappa_l + c_l)P_l^3 - (a_l + 3\gamma_l\kappa_l)\frac{\partial^2 P_l}{\partial x^2} - b_l\frac{\partial^2 P_l}{\partial x\partial t} = 0, \tag{6}$$

and

$$(1 - b_l\kappa_l)\frac{\partial P_l}{\partial t} - (2a_l\kappa_l - b_l\omega_l - \alpha_l + 3\gamma_l\kappa_l^2)\frac{\partial P_l}{\partial x} + \left(3\beta_l + 2\nu_l + \theta_l\right)P_l^2\frac{\partial P_l}{\partial x} + \gamma_l\frac{\partial^3 P_l}{\partial x^3} = 0, \tag{7}$$

for $l = 1, 2$ and $\bar{l} = 3 - l$. Under the traveling wave transformation

$$P_1(x, t) = U_1(\xi), \quad P_2(x, t) = U_2(\xi), \quad \xi = B(x - vt), \tag{8}$$

we have

$$(\omega_l + a_l\kappa_l^2 - b_l\kappa_l\omega_l - \alpha_l\kappa_l + \gamma_l\kappa_l^3)U_l - d_l U_l U_{\bar{l}}^2 - (\beta_l\kappa_l + \theta_l\kappa_l + c_l)U_l^3 - (a_l - b_l v + 3\gamma_l\kappa_l)B^2\frac{d^2U_l}{d\xi^2} = 0, \tag{9}$$

and

$$-\left(v(1 - b_l\kappa_l) + 2a_l\kappa_l - b_l\omega_l - \alpha_l + 3\gamma_l\kappa_l^2\right)B\frac{dU_l}{d\xi} + \left(3\beta_l + 2\nu_l + \theta_l\right)BU_l^2\frac{dU_l}{d\xi} + \gamma_l B^3\frac{d^3U_l}{d\xi^3} = 0. \tag{10}$$

From (10), the third term implies

$$\gamma_l = 0, \tag{11}$$

for $l = 1, 2$ so that Eqs. (9) and (10) reduces to

$$(\omega_l + a_l\kappa_l^2 - b_l\kappa_l\omega_l - \alpha_l\kappa_l)U_l - d_l U_l U_{\bar{l}}^2 - (\beta_l\kappa_l + \theta_l\kappa_l + c_l)U_l^3 - (a_l - b_l v)B^2\frac{d^2U_l}{d\xi^2} = 0, \tag{12}$$

and

$$-(v(1 - b_l\kappa_l) + 2a_l\kappa_l - b_l\omega_l - \alpha_l)B\frac{dU_l}{d\xi} + \left(3\beta_l + 2\nu_l + \theta_l\right)BU_l^2\frac{dU_l}{d\xi} = 0. \tag{13}$$

Therefore, exact solutions for Kerr law nonlinearity in birefringent fibers will exist provided the third order dispersion coefficient is zero. This is an important issue that was earlier reported in 2014 [1]. Next, setting the coefficients of the linearly independent functions, in (10), to zero gives

$$v = \frac{2a_l\kappa_l - b_l\omega_l - \alpha_l}{b_l\kappa_l - 1}, \tag{14}$$

and

$$3\beta_l + 2\nu_l + \theta_l = 0. \tag{15}$$

Eq. (14) is the velocity of the soliton for two components and Eq. (15) represents the constraint condition in order for the soliton to exist. Also, from (14), the following condition is needed:

$$b_l\kappa_l \neq 1. \tag{16}$$

Then, from (14), equating the velocity of the solitons leads to a constraint condition given by

$$(2a_1\kappa_1 - b_1\omega_1 - \alpha_1)(b_2\kappa_2 - 1) = (2a_2\kappa_2 - b_2\omega_2 - \alpha_2)(b_1\kappa_1 - 1). \tag{17}$$

Consequently, the real part Eq. (9) changes to

$$(\omega_1 + a_1\kappa_1^2 - b_1\kappa_1\omega_1 - \alpha_1\kappa_1)U_1 - d_1U_1U_2^2 - (\beta_1\kappa_1 + \theta_1\kappa_1 + c_1)U_1^3 - (a_1 - b_1v)B^2\frac{d^2U_1}{d\xi^2} = 0, \tag{18}$$

$$(\omega_2 + a_2\kappa_2^2 - b_2\kappa_2\omega_2 - \alpha_2\kappa_2)U_2 - d_2U_2U_1^2 - (\beta_2\kappa_2 + \theta_2\kappa_2 + c_2)U_2^3 - (a_2 - b_2v)B^2\frac{d^2U_2}{d\xi^2} = 0. \tag{19}$$

We will now analyze the system of equations (18) and (19) to secure soliton solutions by extended trial equation method in the following subsection.

2.1. Extended trial equation method

For solutions to (18) and (19), the following assumption for the soliton structure is made

$$U_1 = \sum_{i=0}^{\varsigma}\tau_i\Psi^i, \tag{20}$$

$$U_2 = \sum_{i=0}^{\tilde{\varsigma}}\tilde{\tau}_i\Psi^i, \tag{21}$$

where

$$(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} = \frac{\mu_\epsilon\Psi^\epsilon + \cdots + \mu_1\Psi + \mu_0}{\chi_\rho\Psi^\rho + \cdots + \chi_1\Psi + \chi_0}. \tag{22}$$

Using the relations (20)–(22), we can derive the terms U_1'' and U_2'' as follows:

$$U_1'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)}\left(\sum_{i=0}^{\varsigma} i\tau_i\Psi^{i-1}\right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\varsigma} i(i-1)\tau_i\Psi^{i-2}\right), \tag{23}$$

and

$$U_2'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)}\left(\sum_{i=0}^{\tilde{\varsigma}} i\tilde{\tau}_i\Psi^{i-1}\right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\tilde{\varsigma}} i(i-1)\tilde{\tau}_i\Psi^{i-2}\right), \tag{24}$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials of Ψ. We can reduce Eq. (22) to the elementary integral form as follows:

$$\pm(\xi - \xi_0) = \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}d\Psi. \tag{25}$$

We substitute Eqs. (20)–(24) into Eqs. (18) and (19). Then, we use the balance principle and find that

$$\varsigma = \tilde{\varsigma} = \frac{\epsilon - \rho - 2}{2}. \tag{26}$$

When $\epsilon = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (26), we have

$$U_1 = \tau_0 + \tau_1\Psi, \tag{27}$$

$$U_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{28}$$

and

$$U_1'' = \frac{\tau_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{29}$$

$$U_2'' = \frac{\tilde{\tau}_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{30}$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. We substitute Eqs. (27)–(30) into Eqs. (18)–(19) and collect the coefficients of Ψ. Then, we solve the resulting system and obtain the following results:

$$\begin{aligned}
&\mu_1 = \mp \frac{2i\tau_0\mathcal{H}_2\left(\mu_2 B^2\mathcal{H}_1^2 + 2\tau_0^2\chi_0\mathcal{H}_3\right)}{\tilde{\tau}_1 B^2 \mathcal{H}_1^3}, \quad \mu_3 = \mp \frac{2i\tau_0\tilde{\tau}_1\chi_0\mathcal{H}_3}{B^2\mathcal{H}_1\mathcal{H}_2}, \quad \mu_4 = -\frac{\tilde{\tau}_1^2\chi_0\mathcal{H}_3}{2B^2\mathcal{H}_4},\\
&\mu_0 = \mu_0, \quad \mu_2 = \mu_2, \quad \chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad \tau_1 = \pm\frac{i\tilde{\tau}_1\mathcal{H}_1}{\mathcal{H}_2}, \quad \tilde{\tau}_0 = \mp\frac{i\tau_0\mathcal{H}_2}{\mathcal{H}_1},\\
&\omega_1 = \frac{\mu_2 B^2\mathcal{H}_1^2(a_1 - b_1\nu) + \chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1 - \alpha_1) + 3\tau_0^2\mathcal{H}_3(a_1 - b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1 - 1)},\\
&\omega_2 = \frac{\mu_2 B^2(a_2 - b_2\nu)(\mathcal{H}_6 - \mathcal{H}_7) + \chi_0\left[\kappa_2(\alpha_2 - a_2\kappa_2)(\mathcal{H}_6 - \mathcal{H}_7) - 3\tau_0^2\mathcal{H}_3(a_2 - b_2\nu)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2 - 1)},
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\mathcal{H}_1 = \sqrt{a_1c_2 - a_2d_1 - b_1c_2\nu + b_2d_1\nu + \kappa_2(a_1 - b_1\nu)(\beta_2 + \theta_2)},\\
&\mathcal{H}_2 = \sqrt{d_2(a_1 - b_1\nu) - a_2(c_1 + \kappa_1(\beta_1 + \theta_1)) + b_2\nu(c_1 + \kappa_1(\beta_1 + \theta_1))},\\
&\mathcal{H}_3 = -d_1d_2 + (c_1 + \kappa_1(\beta_1 + \theta_1))(c_2 + \kappa_2(\beta_2 + \theta_2)),\\
&\mathcal{H}_4 = a_2c_1 - a_1d_2 - b_2c_1\nu + b_1d_2\nu + \kappa_1(a_2 - b_2\nu)(\beta_1 + \theta_1),\\
&\mathcal{H}_5 = d_1(a_2 - b_2\nu) - a_1(c_2 + \kappa_2(\beta_2 + \theta_2)) + b_1\nu(c_2 + \kappa_2(\beta_2 + \theta_2)),\\
&\mathcal{H}_6 = d_1(a_2 - b_2\nu) + b_1\nu(c_2 + \kappa_2(\beta_2 + \theta_2)), \quad \mathcal{H}_7 = a_1(c_2 + \kappa_2(\beta_2 + \theta_2)).
\end{aligned} \tag{32}$$

Substituting the solution set (31) into Eqs. (22) and (25), we find that

$$\pm(\xi - \xi_0) = W\int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{33}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{34}$$

We integrate Eq. (33) and insert the result into Eqs. (27) and (28). Then, we obtain the exact solutions to Eqs. (18) and (19). Consequently, we have the traveling wave solutions to the coupled NLSE with STD and Kerr law nonlinearity as follows:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, then we obtain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right) - \xi_0}\right\} \times e^{i\left\{-\kappa_1 x + \left(\frac{\mu_2 B^2\mathcal{H}_1^2(a_1 - b_1\nu) + \chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1 - \alpha_1) + 3\tau_0^2\mathcal{H}_3(a_1 - b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1 - 1)}\right)t + \sigma_1\right\}}, \tag{35}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 \pm \frac{\tilde{\tau}_1 W}{B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right) - \xi_0}\right\} \times e^{i\left\{-\kappa_2 x + \left(\frac{\mu_2 B^2(a_2 - b_2\nu)(\mathcal{H}_6 - \mathcal{H}_7) + \chi_0\left[\kappa_2(\alpha_2 - a_2\kappa_2)(\mathcal{H}_6 - \mathcal{H}_7) - 3\tau_0^2\mathcal{H}_3(a_2 - b_2\nu)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2 - 1)}\right)t + \sigma_2\right\}}. \tag{36}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$, then we get

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right)\right]^2}\right\}\times e^{i\left\{-\kappa_1 x+\left(\frac{\mu_2 B^2\mathcal{H}_1^2(a_1-b_1 v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1 v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \tag{37}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{4W^2(\lambda_2-\lambda_1)\tilde{\tau}_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right)\right]^2}\right\}\times e^{i\left\{-\kappa_2 x+\left(\frac{\mu_2 B^2(a_2-b_2 v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2 v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}. \tag{38}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, then we have

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right)\right]-1}\right\}\times e^{i\left\{-\kappa_1 x+\left(\frac{\mu_2 B^2\mathcal{H}_1^2(a_1-b_1 v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1 v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \tag{39}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tilde{\tau}_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right)\right]-1}\right\}\times e^{i\left\{-\kappa_2 x+\left(\frac{\mu_2 B^2(a_2-b_2 v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2 v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}, \tag{40}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right)\right]-1}\right\}\times e^{i\left\{-\kappa_1 x+\left(\frac{\mu_2 B^2\mathcal{H}_1^2(a_1-b_1 v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1 v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \tag{41}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tilde{\tau}_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right)\right]-1}\right\}\times e^{i\left\{-\kappa_2 x+\left(\frac{\mu_2 B^2(a_2-b_2 v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2 v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}. \tag{42}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}\left[B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right]\right)}\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1\nu)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{43}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tilde{\tau}_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}\left[B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right]\right)}\right\}\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2\nu)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2\nu)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}.\tag{44}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}\left[B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right],l\right]}\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1\nu)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{45}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}\left[B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right],l\right]}\right\}\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2\nu)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2\nu)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}},\tag{46}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}.\tag{47}$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0.\tag{48}$$

When $\tau_0=-\tau_1\lambda_1$, $\tilde{\tau}_0=-\tilde{\tau}_1\lambda_1$ and $\xi_0=0$, then we can reduce the solutions (35)–(44) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1W}{B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)}\right\}e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1\nu)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{49}$$

$$r(x,t)=\left\{\pm\frac{\tilde{\tau}_1W}{B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)}\right\}e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2\nu)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2\nu)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}},\tag{50}$$

$$q(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]^2}\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1\nu)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1\nu)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{51}$$

$$r(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tilde{\tau}_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]^2}\right\}\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}, \quad (52)$$

singular soliton solutions

$$q(x,t)=\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \quad (53)$$

$$r(x,t)=\frac{(\lambda_2-\lambda_1)\tilde{\tau}_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]\right\}\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}, \quad (54)$$

and bright soliton solutions

$$q(x,t)=\left\{\frac{A}{C+\cosh\left[D\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]}\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \quad (55)$$

$$r(x,t)=\left\{\frac{\tilde{A}}{C+\cosh\left[D\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]}\right\}\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}}, \quad (56)$$

where

$$A=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2},\quad \tilde{A}=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tilde{\tau}_1}{\lambda_3-\lambda_2},\quad D=\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W},\quad C=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \quad (57)$$

Here, A and $\tilde{A}$ are the amplitudes of the solitons, while D is related to inverse width of the solitons. These solitons exist for $\tau_1<0$ and $\tilde{\tau}_1<0$. Moreover, when $\tau_0=-\tau_1\lambda_2$, $\tilde{\tau}_0=-\tilde{\tau}_1\lambda_2$ and $\xi_0=0$, we can write the Jacobi elliptic function solutions (45) and (46) as follows:

$$q(x,t)=\left\{\frac{A_1}{C_1+\mathrm{sn}^2\left[D_j\left[B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right],\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\}\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}}, \quad (58)$$

$$r(x,t)=\left\{\frac{\tilde{A}_1}{C_1+\mathrm{sn}^2\left[D_j\left[B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right],\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\}$$
$$\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}},\tag{59}$$

where

$$A_1=\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4},\quad \tilde{A}_1=\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4},\quad C_1=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},$$
$$D_j=\frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W},\qquad (j=1,2).\tag{60}$$

Remark 1. When the modulus $l\to 1$, a second form of singular optical soliton solutions are obtained:

$$q(x,t)=\left\{\frac{A_1}{C_1+\tanh^2\left[D_j\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]}\right\}$$
$$\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{61}$$

$$r(x,t)=\left\{\frac{\tilde{A}_1}{C_1+\tanh^2\left[D_j\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]}\right\}$$
$$\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}},\tag{62}$$

where $\lambda_3=\lambda_4$.

Remark 2. However, if $l\to 0$, the following periodic singular solutions emerge

$$q(x,t)=\left\{\frac{A_1}{C_1+\sin^2\left[D_j\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]}\right\}$$
$$\times e^{i\left\{-\kappa_1x+\left(\frac{\mu_2B^2\mathcal{H}_1^2(a_1-b_1v)+\chi_0\left[\kappa_1\mathcal{H}_5(a_1\kappa_1-\alpha_1)+3\tau_0^2\mathcal{H}_3(a_1-b_1v)\right]}{\chi_0\mathcal{H}_5(b_1\kappa_1-1)}\right)t+\sigma_1\right\}},\tag{63}$$

$$r(x,t)=\left\{\frac{\tilde{A}_1}{C_1+\sin^2\left[D_j\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]}\right\}$$
$$\times e^{i\left\{-\kappa_2x+\left(\frac{\mu_2B^2(a_2-b_2v)(\mathcal{H}_6-\mathcal{H}_7)+\chi_0\left[\kappa_2(\alpha_2-a_2\kappa_2)(\mathcal{H}_6-\mathcal{H}_7)-3\tau_0^2\mathcal{H}_3(a_2-b_2v)\right]}{\chi_0\mathcal{H}_1^2(b_2\kappa_2-1)}\right)t+\sigma_2\right\}},\tag{64}$$

where $\lambda_2=\lambda_3$.

3. Parabolic law nonlinearity

For parabolic law nonlinearity, the coupled NLSE in birefringent fibers with GVD and STD is given by [1]

$$iq_t+a_1q_{xx}+b_1q_{xt}+\left(c_1|q|^2+d_1|r|^2\right)q+\left(\delta_1|q|^4+\eta_1|q|^2|r|^2+\zeta_1|r|^4\right)q$$
$$+i\left\{\alpha_1q_x+\beta_1\left(|q|^2q\right)_x+\nu_1\left(|q|^2\right)_xq+\theta_1|q|^2q_x+\gamma_1q_{xxx}\right\}=0,\tag{65}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + \left(c_2|r|^2 + d_2|q|^2\right) r + \left(\delta_2|r|^4 + \eta_2|r|^2|q|^2 + \zeta_2|q|^4\right) q$$
$$+ i\left\{\alpha_2 r_x + \beta_2\left(|r|^2 r\right)_x + \nu_2\left(|r|^2\right)_x r + \theta_2|r|^2 r_x + \gamma_2 r_{xxx}\right\} = 0. \quad (66)$$

In (65) and (66), terms with c_l and δ_l are associated with SPM while d_l, η_l and ζ_l are from XPM for parabolic law nonlinearity. The rest of the terms have the same interpretation as with the Kerr law nonlinearity. Substituting the same hypothesis as in (3) and (4) into (65) and (66), and decomposing into real and imaginary parts yields

$$(\omega_l + a_l\kappa_l^2 - b_l\kappa_l\omega_l - \alpha_l\kappa_l + \gamma_l\kappa_l^3)P_l - d_l P_l P_{\bar{l}}^2 - \delta_l P_l^5 - \eta_l P_l^3 P_{\bar{l}} - \zeta_l P_l P_{\bar{l}}^4 - (\beta_l\kappa_l + \theta_l\kappa_l + c_l)P_l^3$$
$$-(a_l + 3\gamma_l\kappa_l)\frac{\partial^2 P_l}{\partial x^2} - b_l\frac{\partial^2 P_l}{\partial x\partial t} = 0, \quad (67)$$

$$(1 - b_l\kappa_l)\frac{\partial P_l}{\partial t} - (2a_l\kappa_l - b_l\omega_l - \alpha_l + 3\gamma_l\kappa_l^2)\frac{\partial P_l}{\partial x} + (3\beta_l + 2\nu_l + \theta_l)P_l^2\frac{\partial P_l}{\partial x} + \gamma_l\frac{\partial^3 P_l}{\partial x^3} = 0, \quad (68)$$

for $l = 1, 2$ and $\bar{l} = 3 - l$. With travelling wave transformation (8), we have

$$\left(\omega_l + a_l\kappa_l^2 - b_l\kappa_l\omega_l - \alpha_l\kappa_l + \gamma_l\kappa_l^3\right) U_l - d_l U_l U_{\bar{l}}^2 - \delta_l U_l^5 - \eta_l U_l^3 U_{\bar{l}} - \zeta_l U_l U_{\bar{l}}^4 - (\beta_l\kappa_l + \theta_l\kappa_l + c_l)U_l^3$$
$$-(a_l - b_l v + 3\gamma_l\kappa_l)B^2\frac{d^2 U_l}{d\xi^2} = 0, \quad (69)$$

$$-\left(v(1 - b_l\kappa_l) + 2a_l\kappa_l - b_l\omega_l - \alpha_l + 3\gamma_l\kappa_l^2\right) B\frac{dU_l}{d\xi} + \left(3\beta_l + 2\nu_l + \theta_l\right) BU_l^2\frac{dU_l}{d\xi} + \gamma_l B^3\frac{d^3 U_l}{d\xi^3} = 0. \quad (70)$$

From (70), once again $\gamma_l = 0$. The speed v of the soliton from the linearly independent functions is also given by (14). The constraint conditions (15) and (16) also fall out from these linearly independent functions. Consequently, the real part Eq. (69) changes to

$$(\omega_1 + a_1\kappa_1^2 - b_1\kappa_1\omega_1 - \alpha_1\kappa_1)U_1 - d_1 U_1 U_2^2 - \delta_1 U_1^5 - \eta_1 U_1^3 U_2 - \zeta_1 U_1 U_2^4 - (\beta_1\kappa_1 + \theta_1\kappa_1 + c_1)U_1^3$$
$$-(a_1 - b_1 v)B^2\frac{d^2 U_1}{d\xi^2} = 0, \quad (71)$$

$$\left(\omega_2 + a_2\kappa_2^2 - b_2\kappa_2\omega_2 - \alpha_2\kappa_2\right) U_2 - d_2 U_2 U_1^2 - \delta_2 U_2^5 - \eta_2 U_2^3 U_1 - \zeta_2 U_2 U_1^4 - (\beta_2\kappa_2 + \theta_2\kappa_2 + c_2)U_2^3$$
$$-(a_2 - b_2 v)B^2\frac{d^2 U_2}{d\xi^2} = 0. \quad (72)$$

We will now analyze the system of equations (71) and (72) to obtain soliton solutions by the extended trial equation method in the next subsection.

3.1. Extended trial equation method

Balancing U_l'' with U_l^5 in Eqs. (71) and (72), we find that

$$N + 2 = 5N \Leftrightarrow 2 = 4N \Leftrightarrow N = \frac{1}{2}. \quad (73)$$

When $\eta_l = 0$, in order to obtain an analytic solution we use the transformation

$$U_1 = V_1^{(1/2)} = V_2^{(1/2)} = U_2, \quad (74)$$

that will reduce Eqs. (71) and (72) into the system of equations

$$4(\omega_1 + a_1\kappa_1^2 - b_1\kappa_1\omega_1 - \alpha_1\kappa_1)V_1^2 - 4(d_1 + \beta_1\kappa_1 + \theta_1\kappa_1 + c_1)V_1^3 - 4(\delta_1 + \zeta_1)V_1^4 + (a_1 - b_1 v)B^2\left\{\left(V_1'\right)^2 - 2V_1 V_1''\right\} = 0, \quad (75)$$

$$4\left(\omega_2 + a_2\kappa_2^2 - b_2\kappa_2\omega_2 - \alpha_2\kappa_2\right) V_2^2 - 4(d_2 + \beta_2\kappa_2 + \theta_2\kappa_2 + c_2)V_2^3 - 4(\delta_2 + \zeta_2)V_2^4 + (a_2 - b_2 v)B^2\left\{\left(V_2'\right)^2 - 2V_2 V_2''\right\} = 0. \quad (76)$$

We substitute Eqs. (20)–(24) into Eqs. (75) and (76). Then, we balance the order of $V_l V_l''$ and V_l^4 in Eqs. (75) and (76) and find that

$$\varsigma = \tilde{\varsigma} = \frac{\epsilon - \rho - 2}{2}. \tag{77}$$

When $\epsilon = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (77), Eqs. (75) and (76) have the solutions in the form

$$V_1(\xi) = \tau_0 + \tau_1 \Psi, \tag{78}$$

$$V_2(\xi) = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{79}$$

where τ_i, $\tilde{\tau}_i$ (i=0, 1) are constants to be determined later. We substitute Eqs. (78)–(79) into Eqs. (75)–(76) and collect the coefficients of Ψ. Then, we solve the resulting system and find the following results:

$$\begin{aligned}
\mu_0 &= \frac{\tau_0\tilde{\tau}_0\left(-3\tau_1^3\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+3\tau_0^2\chi_0\tilde{\tau}_1^3\mathcal{H}_4(a_1-b_1\nu)+\tau_1^2\tilde{\tau}_0\tilde{\tau}_1\mathcal{H}_1+\tau_0\tau_1\tilde{\tau}_1^2\mathcal{H}_2\right)}{2\tau_1^2\tilde{\tau}_1^2B^2(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(a_1-b_1\nu)(b_2\nu-a_2)},\\
\mu_1 &= \frac{\chi_0(\tilde{\tau}_1\mathcal{H}_4\mathcal{H}_6(a_1-b_1\nu)+\tau_1\mathcal{H}_3\mathcal{H}_5(b_2\nu-a_2))+2\mu_2\tau_1\tilde{\tau}_1B^2\left(\tau_1^2\tilde{\tau}_0^2-\tau_0^2\tilde{\tau}_1^2\right)(a_1-b_1\nu)(b_2\nu-a_2)}{2\tau_1^2\tilde{\tau}_1^2B^2(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(a_1-b_1\nu)(b_2\nu-a_2)},\\
\mu_3 &= \frac{2\chi_0\left(\tau_1^2\tilde{\tau}_0\mathcal{H}_3(a_2-b_2\nu)+\tau_0\tilde{\tau}_1^2\mathcal{H}_4(b_1\nu-a_1)\right)}{B^2(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(a_1-b_1\nu)(b_2\nu-a_2)},\quad \mu_4=\frac{\tau_1\chi_0\tilde{\tau}_1(\tilde{\tau}_1\mathcal{H}_4(b_1\nu-a_1)+\tau_1\mathcal{H}_3(a_2-b_2\nu))}{2B^2(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(a_1-b_1\nu)(b_2\nu-a_2)},\\
\omega_1 &= \frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)},\\
\omega_2 &= \frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)},\\
\delta_1 &= \frac{-8\zeta_1\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)+\tau_1\tilde{\tau}_1(a_2-b_2\nu)(8\zeta_1\tau_0+3\mathcal{H}_3)-3\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)}{8\tau_1(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(a_2-b_2\nu)},\\
\delta_2 &= \frac{\tilde{\tau}_1(a_1-b_1\nu)(\tau_1(8\zeta_2\tilde{\tau}_0+3\mathcal{H}_4)-8\zeta_2\tau_0\tilde{\tau}_1)-3a_2\tau_1^2\mathcal{H}_3+3b_2\tau_1^2\nu\mathcal{H}_3}{8\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(a_1-b_1\nu)},\\
\mu_2 &= \mu_2,\quad \chi_0=\chi_0,\quad \tau_0=\tau_0,\quad \tau_1=\tau_1,\quad \tilde{\tau}_0=\tilde{\tau}_0,\quad \tilde{\tau}_1=\tilde{\tau}_1,
\end{aligned} \tag{80}$$

where

$$\begin{aligned}
\mathcal{H}_1 &= (a_2-b_2\nu)\left(-2\mu_2B^2(a_1-b_1\nu)-3\tau_0\chi_0\left(\kappa_1\left(\beta_1+\theta_1\right)+c_1+d_1\right)\right)-\chi_0\tilde{\tau}_0(a_1-b_1\nu)\left(\kappa_2\left(\beta_2+\theta_2\right)+c_2+d_2\right),\\
\mathcal{H}_2 &= 3\chi_0\tilde{\tau}_0(a_1-b_1\nu)\left(\kappa_2\left(\beta_2+\theta_2\right)+c_2+d_2\right)+(a_2-b_2\nu)\left(2\mu_2B^2(a_1-b_1\nu)+\tau_0\chi_0\left(\kappa_1\left(\beta_1+\theta_1\right)+c_1+d_1\right)\right),\\
\mathcal{H}_3 &= \kappa_1\left(\beta_1+\theta_1\right)+c_1+d_1,\quad \mathcal{H}_4=\kappa_2\left(\beta_2+\theta_2\right)+c_2+d_2,\quad \mathcal{H}_5=3\tau_1^3\tilde{\tau}_0^3+3\tau_0\tau_1^2\tilde{\tau}_1\tilde{\tau}_0^2+3\tau_0^2\tau_1\tilde{\tau}_1^2\tilde{\tau}_0-\tau_0^3\tilde{\tau}_1^3,\\
\mathcal{H}_6 &= -\tau_1^3\tilde{\tau}_0^3+3\tau_0\tau_1^2\tilde{\tau}_1\tilde{\tau}_0^2+3\tau_0^2\tau_1\tilde{\tau}_1^2\tilde{\tau}_0+3\tau_0^3\tilde{\tau}_1^3,\quad \mathcal{H}_9=\tau_0\tau_1\tilde{\tau}_1(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right),\\
\mathcal{H}_7 &= \tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(b_1\mu_2\nu B^2-2\chi_0\left(2\alpha_1\kappa_1+3\tau_0\left(\kappa_1\left(\beta_1+\theta_1\right)+c_1+d_1\right)\right)\right),\\
\mathcal{H}_8 &= \tau_0\tau_1\tilde{\tau}_1(b_2\nu-a_2)\left(b_1\mu_2\nu B^2-\chi_0\left(4\alpha_1\kappa_1+3\tau_0\left(\kappa_1\left(\beta_1+\theta_1\right)+c_1+d_1\right)\right)\right),\\
\mathcal{H}_{10} &= \tau_1\tilde{\tau}_0\tilde{\tau}_1(a_1-b_1\nu)\left(\chi_0\left(3\tilde{\tau}_0\left(\kappa_2\left(\beta_2+\theta_2\right)+c_2+d_2\right)+4\alpha_2\kappa_2\right)+a_2\left(\mu_2B^2-4\kappa_2^2\chi_0\right)-b_2\mu_2\nu B^2\right),\\
\mathcal{H}_{11} &= \tau_0\tilde{\tau}_1^2(a_1-b_1\nu)\left(2\chi_0\left(3\tilde{\tau}_0\left(\kappa_2\left(\beta_2+\theta_2\right)+c_2+d_2\right)+2\alpha_2\kappa_2\right)+a_2\left(\mu_2B^2-4\kappa_2^2\chi_0\right)-b_2\mu_2\nu B^2\right).
\end{aligned} \tag{81}$$

Substituting the solution set (80) into Eqs. (22) and (25), we find that

$$\pm(\xi-\xi_0) = W\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{82}$$

where

$$\Lambda(\Psi) = \Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},\quad W=\sqrt{\frac{\chi_0}{\mu_4}}. \tag{83}$$

Integrating Eq. (82) and inserting the result into Eqs. (78) and (79), we can write the solution formulas of Eqs. (75) and (76). Then, using the transformation $U_l = V_l^{(1/2)}$, we can obtain solutions of Eqs. (71) and (72) as follows:

$$U_1(\xi) = \left\{\tau_0+\tau_1\Psi\right\}^{(1/2)}, \tag{84}$$

$$U_2(\xi) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1 \Psi\right\}^{(1/2)}. \tag{85}$$

Consequently, we have the following exact solutions to the coupled NLSE in birefringent fibers with GVD and STD (65) and (66):

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, then we obtain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right) - \xi_0}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2 - b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1 - b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1 - 1)(a_2 - b_2\nu)}\right)t + \sigma_1\right\}}, \tag{86}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 \pm \frac{\tilde{\tau}_1 W}{B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right) - \xi_0}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2 - b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2 - 1)(a_1 - b_1\nu)}\right)t + \sigma_2\right\}}. \tag{87}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, then we get

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4W^2(\lambda_2 - \lambda_1)\tau_1}{4W^2 - \left[(\lambda_1 - \lambda_2)\left(B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right) - \xi_0\right)\right]^2}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2 - b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1 - b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1 - 1)(a_2 - b_2\nu)}\right)t + \sigma_1\right\}}, \tag{88}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4W^2(\lambda_2 - \lambda_1)\tilde{\tau}_1}{4W^2 - \left[(\lambda_1 - \lambda_2)\left(B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right) - \xi_0\right)\right]^2}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2 - b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2 - 1)(a_1 - b_1\nu)}\right)t + \sigma_2\right\}}. \tag{89}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, then we have

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2 - \lambda_1)\tau_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{W}\left(B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right) - \xi_0\right)\right] - 1}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2 - b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1 - b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1 - 1)(a_2 - b_2\nu)}\right)t + \sigma_1\right\}}, \tag{90}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{(\lambda_2 - \lambda_1)\tilde{\tau}_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{W}\left(B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right) - \xi_0\right)\right] - 1}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2 - b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2 - 1)(a_1 - b_1\nu)}\right)t + \sigma_2\right\}}, \tag{91}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right)\right]-1}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{92}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tilde{\tau}_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right)\right]-1}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}. \tag{93}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}\left[B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right]\right)}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{94}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tilde{\tau}_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}\left[B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right]\right)}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}. \tag{95}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}\left[B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)-\xi_0\right],l\right]}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{96}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}\left[B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)-\xi_0\right],l\right]}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}, \tag{97}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{98}$$

Also, λ_i ($i=1,\ldots,4$) are the roots of the polynomial equation

$$\Lambda(\Psi)=0. \tag{99}$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\xi_0 = 0$, then we can reduce the solutions (86)-(95) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1 W}{B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)}\right\}^{(1/2)} e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{100}$$

$$r(x,t)=\left\{\pm\frac{\tilde{\tau}_1 W}{B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)}\right\}^{(1/2)} e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}, \tag{101}$$

$$q(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]^2}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{102}$$

$$r(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tilde{\tau}_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]^2}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}, \tag{103}$$

singular soliton solutions

$$q(x,t)=\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{104}$$

$$r(x,t)=\frac{(\lambda_2-\lambda_1)\tilde{\tau}_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}, \tag{105}$$

and bright soliton solutions

$$q(x,t)=\left\{\frac{A}{C+\cosh\left[D\left(B\left(x-\frac{2a_1\kappa_1-b_1\omega_1-\alpha_1}{b_1\kappa_1-1}t\right)\right)\right]}\right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x+\left(\frac{a_1\left(\mathcal{H}_9-\tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2B^2-4\kappa_1^2\chi_0\right)\right)+3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu)+\mathcal{H}_7+\mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0-\tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t+\sigma_1\right\}}, \tag{106}$$

$$r(x,t)=\left\{\frac{\tilde{A}}{C+\cosh\left[D\left(B\left(x-\frac{2a_2\kappa_2-b_2\omega_2-\alpha_2}{b_2\kappa_2-1}t\right)\right)\right]}\right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x+\left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu)+\mathcal{H}_{10}-\mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1-\tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t+\sigma_2\right\}}, \tag{107}$$

where

$$A=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2},\quad \tilde{A}=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tilde{\tau}_1}{\lambda_3-\lambda_2},\quad D=\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W},\quad C=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \tag{108}$$

Here, A and $\tilde{A}$ are the amplitudes of the solitons, while D is related to inverse width of the solitons. These solitons exist for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_2$ and $\xi_0 = 0$, we can write the Jacobi elliptic function solutions (96) and (97) as follows:

$$q(x,t) = \left\{ \frac{A_1}{C_1 + \mathrm{sn}^2\left[D_j\left[B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right)\right], \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t + \sigma_1\right\}}, \quad (109)$$

$$r(x,t) = \left\{ \frac{\tilde{A}_1}{C_1 + \mathrm{sn}^2\left[D_j\left[B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right)\right], \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t + \sigma_2\right\}}, \quad (110)$$

where

$$A_1 = \frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}, \quad \tilde{A}_1 = \frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4},$$
$$C_1 = \frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4}, \quad D_j = \frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}, \; (j=1,2). \quad (111)$$

Remark 3. When the modulus $l \to 1$, a second form of singular optical soliton solutions fall out:

$$q(x,t) = \left\{ \frac{A_1}{C_1 + \tanh^2\left[D_j\left(B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right)\right)\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t + \sigma_1\right\}}, \quad (112)$$

$$r(x,t) = \left\{ \frac{\tilde{A}_1}{C_1 + \tanh^2\left[D_j\left(B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right)\right)\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t + \sigma_2\right\}}, \quad (113)$$

where $\lambda_3 = \lambda_4$.

Remark 4. However, if $l \to 0$, periodic singular solutions are as listed below:

$$q(x,t) = \left\{ \frac{A_1}{C_1 + \sin^2\left[D_j\left(B\left(x - \frac{2a_1\kappa_1 - b_1\omega_1 - \alpha_1}{b_1\kappa_1 - 1}t\right)\right)\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_1 x + \left(\frac{a_1\left(\mathcal{H}_9 - \tau_1^2\tilde{\tau}_0(a_2-b_2\nu)\left(\mu_2 B^2 - 4\kappa_1^2\chi_0\right)\right) + 3\tau_0^2\chi_0\tilde{\tau}_1^2\mathcal{H}_4(a_1-b_1\nu) + \mathcal{H}_7 + \mathcal{H}_8}{4\tau_1\chi_0(\tau_1\tilde{\tau}_0 - \tau_0\tilde{\tau}_1)(b_1\kappa_1-1)(a_2-b_2\nu)}\right)t + \sigma_1\right\}}, \quad (114)$$

$$r(x,t) = \left\{ \frac{\tilde{A}_1}{C_1 + \sin^2\left[D_j\left(B\left(x - \frac{2a_2\kappa_2 - b_2\omega_2 - \alpha_2}{b_2\kappa_2 - 1}t\right)\right)\right]} \right\}^{(1/2)} \times e^{i\left\{-\kappa_2 x + \left(\frac{3\tau_1^2\chi_0\tilde{\tau}_0^2\mathcal{H}_3(a_2-b_2\nu) + \mathcal{H}_{10} - \mathcal{H}_{11}}{4\chi_0\tilde{\tau}_1(\tau_0\tilde{\tau}_1 - \tau_1\tilde{\tau}_0)(b_2\kappa_2-1)(a_1-b_1\nu)}\right)t + \sigma_2\right\}}, \quad (115)$$

where $\lambda_2 = \lambda_3$.

4. Conclusions

This paper carried out a comprehensive study of optical solitons in birefringent fibers by the aid of extended trial function method that is indeed a powerful integration tool. This algorithm led to bright and singular optical soliton solutions for birefringent fibers with Kerr and parabolic law nonlinearity. There are constraint conditions that naturally emerged from the algebraic structure of solutions and these conditions are indeed mandated for the soliton solutions to exist. There are several other solutions that fell out from the parameter connections. These are singular periodic solutions and doubly periodic solutions.

This paper stands on a strong footing for additional investigation in this subject matter. The FWM terms are to be included. Later, the study will be extended to DWDM systems with FWM and other perturbation terms. Other integration tools will be applied to obtain additional solutions. These are Lie symmetry analysis, mapping method, Kudryashov's method and several others. Those results will be disseminated soon.

Conflict of interest

The authors declare that there is no conflict of interest.

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