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Small inductive dimension and Alexandroff topological spaces

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ABSTRACT

Alexandroff spaces include finite spaces and have a wide range of applications in many areas such as computer graphics and image analysis. We give results on small inductive dimension of Alexandroff T_0-spaces. Particularly, we characterize the small inductive dimension through connected tuples and study standard properties of Dimension Theory using this characterization. Also, we investigate the relations between the small inductive dimension, the large inductive dimension, and the covering dimension in the class of all Alexandroff T_0-spaces. We finally put questions on dimension of Alexandroff T_0-spaces.

1. Introduction and preliminaries

By an *Alexandroff space* (see [1]) we mean a space such that every point has a minimal open neighborhood. This is also equivalent to the fact that the intersection of every family of open sets is open. For every Alexandroff space X and $x \in X$ we denote by $\mathbf{U}_x$ the smallest open set of X containing the point x.

A topological space X is called T_0*-space* if for every pair of distinct points of X, at least one of them has an open neighborhood not containing the other.

Specialization order on Alexandroff spaces is an additional algebraic structure. In [9] it was shown that poset dimension h coincides with small inductive dimension of Alexandroff T_0-spaces. Thus dimension of Alexandroff T_0-spaces can be calculated formally algebraically.

The presented paper is continuation of this approach to study dimensions of Alexandroff T_0-spaces. In Section 2 we give subspace, addition, sum, and product theorems for the height h of Alexandroff spaces. In Section 3 we prove addition, decomposition, and structure theorems concerning the small inductive

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dimension for Alexandroff T_0-spaces. We use structure characterization of ind to sharpen results on sum and product theorems from [3] without restriction on number of factors. Relations of the structure of sum and finite product and factors are given. In Section 4 we investigate the relations between the small inductive dimension, large inductive dimension and covering dimension in the class of all Alexandroff T_0-spaces. Finally, in Section 5 we present some questions.

In what follows we consider two symbols, "-1" and "∞", for which we suppose that:

(1) $-1 < k < \infty$ for every $k \in \{0, 1, \ldots\}$.
(2) $\infty + k = k + \infty = \infty$ and $-1 + k = k + (-1) = k$ for every $k \in \{0, 1, 2, \ldots\} \cup \{-1, \infty\}$.

Also, if X is a topological space and Q a subset of X, then by $\mathrm{Cl}_X(Q)$ and $\mathrm{Bd}_X(Q)$ we denote the closure and the boundary of Q in X, respectively.

2. Height of Alexandroff spaces

Definition 2.1. Let X be an Alexandroff space. Any 1-tuple (α), where $\alpha \in X$, is called *connected*. Also, an $(m+1)$-tuple $(\alpha_0, \ldots, \alpha_m)$, $m \geqslant 1$, of distinct elements of X is called *connected* if and only if $\alpha_{i-1} \in \mathrm{Cl}_X(\{\alpha_i\})$, for every $i \in \{1, \ldots, m\}$.

The following two propositions can be easily proved.

Proposition 2.2. *Let X be an Alexandroff space and $\alpha_0, \ldots, \alpha_m$, $m \geqslant 1$, distinct elements of X. The following statements are equivalent:*

(1) *The $(m+1)$-tuple $(\alpha_0, \ldots, \alpha_m)$ is connected.*
(2) $\mathrm{Cl}_X(\{\alpha_{i-1}\}) \subseteq \mathrm{Cl}_X(\{\alpha_i\})$, *for every* $i \in \{1, \ldots, m\}$.
(3) $\alpha_i \in \mathbf{U}_{\alpha_{i-1}}$, *for every* $i \in \{1, \ldots, m\}$.

Proposition 2.3. *Let X be an Alexandroff space and Y a subspace of X. If an $(m+1)$-tuple $(\alpha_0, \ldots, \alpha_m)$ of distinct elements of Y is connected in Y, then this tuple is connected in X.*

Definition 2.4. The *height of an Alexandroff space* X, denoted by $h(X)$, is defined as follows:

(i) $h(X) = -1$ if and only if $X = \emptyset$.
(ii) $h(X) = k$, where $k \in \{0, 1, \ldots\}$, if and only if there exists a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X and any $(k+2)$-tuple of distinct elements of X is non-connected.
(iii) $h(X) = \infty$ if and only if for every $k \in \{0, 1, \ldots\}$ there exists a connected $(k+1)$-tuple of distinct elements of X.

Definition 2.5. The *height of an Alexandroff space X at the point $x \in X$*, denoted by $h_x(X)$, is defined as follows:

(i) $h_x(X) = k$, where $k \in \{0, 1, \ldots\}$, if there exists a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X with $\{\alpha_0, \ldots, \alpha_k\} \cap \mathbf{U}_x = \{\alpha_k\}$ and any $(k+2)$-tuple $(\alpha'_0, \ldots, \alpha'_{k+1})$ of distinct elements of X with $\{\alpha'_0, \ldots, \alpha'_{k+1}\} \cap \mathbf{U}_x = \{\alpha'_{k+1}\}$ is non-connected.
(ii) $h_x(X) = \infty$ if and only if for every $k \in \{0, 1, \ldots\}$ there exists a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X with $\{\alpha_0, \ldots, \alpha_k\} \cap \mathbf{U}_x = \{\alpha_k\}$.

Proposition 2.6. *Let X be an Alexandroff T_0-space with $h(X) < \infty$. Then,*

$$h(X) = \max\{h_x(X):\ x \in X\}.$$

Proof. The inequality $\max\{h_x(X):\ x \in X\} \leqslant h(X)$ obviously holds. We prove that $h(X) \leqslant \max\{h_x(X):\ x \in X\}$. Let $h(X) = k$, where $k \in \{0, 1, \ldots\}$. There exists a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X. We have $\alpha_k \in \mathbf{U}_{\alpha_k} \subseteq \cdots \subseteq \mathbf{U}_{\alpha_0}$. Therefore, since the space X is a T_0-space, $\{\alpha_0, \ldots, \alpha_k\} \cap \mathbf{U}_{\alpha_k} = \{\alpha_k\}$. This means that $h_{\alpha_k}(X) \geqslant k$ and, consequently, $\max\{h_x(X):\ x \in X\} \geqslant k$. Thus, $\max\{h_x(X):\ x \in X\} \geqslant h(X)$. □

Proposition 2.7. *Let X be an Alexandroff T_0-space with $h(X) < \infty$. Then, the set $D = \{x \in X:\ \{x\}$ is open$\}$ is a dense subset of X.*

Proof. Let U be a non-empty open subset of X. It suffices to prove that $U \cap D \neq \emptyset$. Since $U \neq \emptyset$, there exists $\alpha_0 \in U$. If $\{\alpha_0\}$ is open, then $\alpha_0 \in U \cap D$ and, therefore, we are done. Suppose that $\mathbf{U}_{\alpha_0} \neq \{\alpha_0\}$. Let $\alpha_1 \in \mathbf{U}_{\alpha_0} \setminus \{\alpha_0\}$. Then, $\mathbf{U}_{\alpha_1} \subseteq \mathbf{U}_{\alpha_0}$ and the 2-tuple (α_0, α_1) of distinct elements of X is connected. Since the space X is T_0, we have $\alpha_0 \notin \mathbf{U}_{\alpha_1}$. If $\{\alpha_1\}$ is open, then we are done. Suppose that $\mathbf{U}_{\alpha_1} \neq \{\alpha_1\}$. Let $\alpha_2 \in \mathbf{U}_{\alpha_1} \setminus \{\alpha_1\}$. Then, $\mathbf{U}_{\alpha_2} \subseteq \mathbf{U}_{\alpha_1} \subseteq \mathbf{U}_{\alpha_0}$ and the 3-tuple $(\alpha_0, \alpha_1, \alpha_2)$ of distinct elements of X is connected. Since the space X is T_0, we have $\alpha_1 \notin \mathbf{U}_{\alpha_2}$. Continuing in the same manner, for every $k \in \{0, 1, \ldots\}$, we construct a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X. Therefore, $h(X) = \infty$ which is a contradiction. Thus, there exists $k \in \{0, 1, \ldots\}$ such that $\alpha_k \in U \cap D$. This means that $U \cap D \neq \emptyset$. □

Proposition 2.8. *Let X be an Alexandroff T_0-space with $h(X) < \infty$. Then, there exists $x_0 \in X$ with $\{x_0\}$ open subset of X such that $h(X) = h_{x_0}(X)$.*

Proof. Let $h(X) = k$, where $k \in \{0, 1, \ldots\}$. Then, there exists a connected $(k+1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X. Since the set of all open singletons is dense (see Proposition 2.7), there exists a point $\alpha_{k+1} \in X$ such that the singleton $\{\alpha_{k+1}\}$ is an open set and $\alpha_{k+1} \in \mathbf{U}_{\alpha_k}$. We observe that if $\alpha_{k+1} = \alpha_i$ for some $i \in \{0, \ldots, k-1\}$, then $\mathbf{U}_{\alpha_i} = \{\alpha_i\}$ which is a contradiction with $\alpha_k \in \mathbf{U}_{\alpha_i}$. Thus, $\alpha_{k+1} \notin \{\alpha_0, \ldots, \alpha_{k-1}\}$. Also, if $\alpha_{k+1} \neq \alpha_k$, then the $(k+2)$-tuple $(\alpha_0, \ldots, \alpha_k, \alpha_{k+1})$ of distinct elements of X is connected which is a contradiction. Therefore, $\alpha_{k+1} = \alpha_k$ and $\mathbf{U}_{\alpha_k} = \{\alpha_k\}$. Since $(\alpha_0, \ldots, \alpha_k)$ is a connected $(k+1)$-tuple of distinct elements of X with $\{\alpha_0, \ldots, \alpha_k\} \cap \mathbf{U}_{\alpha_k} = \{\alpha_k\}$, we have $h_{\alpha_k}(X) \geqslant k$. Moreover, $h_{\alpha_k}(X) \leqslant h(X) = k$. Setting $x_0 = \alpha_k$, we have that $\{x_0\}$ is an open subset of X and $h(X) = h_{x_0}(X)$. □

By Propositions 2.6 and 2.8 we have the following corollary.

Corollary 2.9. *Let X be an Alexandroff T_0-space with $h(X) < \infty$. Then,*

$$h(X) = \max\{h_x(X):\ x \in X \text{ with } \{x\} \text{ open subset of } X\}.$$

The following proposition can be easily proved.

Proposition 2.10 *(Subspace Theorem for h). Let Y be a subset of an Alexandroff space X. Then, $h(Y) \leqslant h(X)$.*

Proposition 2.11 *(Addition Theorem for h). Let X be an Alexandroff space and X_1, X_2 subsets of X. Then, $h(X_1 \cup X_2) \leqslant h(X_1) + h(X_2) + 1$.*

Proof. In the case where $h(X_1) = \infty$ or $h(X_2) = \infty$ the inequality is true. Let $h(X_1) = m_1$ and $h(X_2) = m_2$, where $m_1, m_2 \in \{0, 1, \ldots\}$. We prove that $h(X_1 \cup X_2) \leqslant m_1 + m_2 + 1$. It suffices to prove that any $(m_1 + m_2 + 3)$-tuple of distinct elements of $X_1 \cup X_2$ is non-connected. Indeed, let $(\alpha_0, \ldots, \alpha_{m_1+m_2+2})$ be a connected $(m_1 + m_2 + 3)$-tuple of distinct elements of $X_1 \cup X_2$. Consider the following two cases:

Case 1. There exist $m_1 + 2$ distinct elements

$$i_0, \ldots, i_{m_1}, i_{m_1+1} \in \{0, \ldots, m_1 + m_2 + 2\}$$

such that $i_0 < \cdots < i_{m_1} < i_{m_1+1}$ and $\{\alpha_{i_0}, \ldots, \alpha_{i_{m_1}}, \alpha_{i_{m_1+1}}\} \subseteq X_1$. In this case the $(m_1 + 2)$-tuple $(\alpha_{i_0}, \ldots, \alpha_{i_{m_1}}, \alpha_{i_{m_1+1}})$ of distinct elements of X_1 is connected. This contradicts the equality $h(X_1) = m_1$.

Case 2. For every $m_1 + 2$ distinct elements

$$i_0, \ldots, i_{m_1}, i_{m_1+1} \in \{0, \ldots, m_1 + m_2 + 2\}$$

we have $\{\alpha_{i_0}, \ldots, \alpha_{i_{m_1}}, \alpha_{i_{m_1+1}}\} \nsubseteq X_1$. In this case there exist t distinct elements

$$i_0, \ldots, i_t \in \{0, \ldots, m_1 + m_2 + 2\},$$

where $t > m_2$ such that $i_0 < \cdots < i_t$ and $\{\alpha_{i_0}, \ldots, \alpha_{i_t}\} \subseteq X_2$. Therefore, the $(t + 1)$-tuple $(\alpha_{i_0}, \ldots, \alpha_{i_t})$ of distinct elements of X_2 is connected. This contradicts the equality $h(X_2) = m_2$.

Thus, $h(X_1 \cup X_2) \leqslant m_1 + m_2 + 1$. □

Proposition 2.12 *(Sum Theorem for h). Let X be an Alexandroff space and X_λ, $\lambda \in \Lambda$, closed subspaces of X. Then,*

$$h\Big(\bigcup\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \sup\big\{h(X_\lambda)\colon\ \lambda \in \Lambda\big\}. \tag{2.1}$$

Proof. If $h(X_\lambda) = \infty$ for some $\lambda \in \Lambda$, then $h(\bigcup\{X_\lambda\colon\ \lambda \in \Lambda\}) = \infty$. Let $h(X_\lambda) \in \{0, 1, \ldots\}$ for every $\lambda \in \Lambda$, $\sup\{h(X_\lambda)\colon\ \lambda \in \Lambda\} \in \{0, 1, \ldots\}$, and $(\alpha_0, \ldots, \alpha_t)$ be a connected $(t + 1)$-tuple of distinct elements of $\bigcup\{X_\lambda\colon\ \lambda \in \Lambda\}$. Then,

$$\alpha_0 \in \mathrm{Cl}_X\big(\{\alpha_1\}\big), \qquad \ldots, \qquad \alpha_{t-1} \in \mathrm{Cl}_X\big(\{\alpha_t\}\big).$$

We prove that $\{\alpha_0, \ldots, \alpha_t\} \subseteq X_\lambda$ for some $\lambda \in \Lambda$. Indeed, let α_{λ_0} and α_{λ_1}, where $\lambda_0, \lambda_1 \in \{0, \ldots, t\}$ and $\lambda_0 < \lambda_1$, be two distinct elements of the union $\bigcup\{X_\lambda\colon\ \lambda \in \lambda\}$ such that

$$\alpha_{\lambda_0} \in X_{\lambda_0}, \qquad \alpha_{\lambda_0} \notin X_\lambda \quad \text{for every } \lambda \neq \lambda_0$$

and

$$\alpha_{\lambda_1} \in X_{\lambda_1}, \qquad \alpha_{\lambda_1} \notin X_\lambda \quad \text{for every } \lambda \neq \lambda_1.$$

Then, $\alpha_{\lambda_0} \in \mathrm{Cl}_X(\{\alpha_{\lambda_1}\})$. Since $\alpha_{\lambda_1} \in X_{\lambda_1}$, we have $\mathrm{Cl}_X(\{\alpha_{\lambda_1}\}) \subseteq X_{\lambda_1}$ and, therefore, $\alpha_{\lambda_0} \in X_{\lambda_1}$ which is a contradiction. Thus, there exists $\lambda' \in \Lambda$ such that $\{\alpha_0, \ldots, \alpha_t\} \subseteq X_{\lambda'}$. This means that

$$h\Big(\bigcup\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) \leqslant h(X_{\lambda'}) \leqslant \sup\big\{h(X_\lambda)\colon\ \lambda \in \Lambda\big\}.$$

Also, $\sup\{h(X_\lambda)\colon\ \lambda\in\Lambda\}\leqslant h(\bigcup\{X_\lambda\colon\ \lambda\in\Lambda\})$. So the relation (2.1) is true. Finally, if $h(X_\lambda)\in\{0,1,\ldots\}$ for every $\lambda\in\Lambda$ and $\sup\{h(X_\lambda)\colon\ \lambda\in\Lambda\}=\infty$, then $h(\bigcup\{X_\lambda\colon\ \lambda\in\Lambda\})=\sup\{h(X_\lambda)\colon\ \lambda\in\Lambda\}=\infty$. □

Proposition 2.13 *(Finite Product Theorem for h). For Alexandroff* T_0*-spaces* X *and* Y *we have* $h(X\times Y)=h(X)+h(Y)$.

Proof. If $h(X)=\infty$ or $h(Y)=\infty$, then we observe that $h(X\times Y)=\infty$. Let $h(X)=m_1$ and $h(Y)=m_2$, where $m_1,m_2\in\{0,1,\ldots\}$. We prove that $h(X\times Y)=m_1+m_2$. There exist a connected (m_1+1)-tuple $(\alpha_0,\ldots,\alpha_{m_1})$ of distinct elements of X and a connected (m_2+1)-tuple $(\beta_0,\ldots,\beta_{m_2})$ of distinct elements of Y. We observe that the (m_1+m_2+1)-tuple

$$\big((\alpha_0,\beta_0),(\alpha_1,\beta_0),\ldots,(\alpha_{m_1},\beta_0),(\alpha_{m_1},\beta_1),(\alpha_{m_1},\beta_2),\ldots,(\alpha_{m_1},\beta_{m_2})\big)$$

of elements of $X\times Y$ is connected.

We prove that any (m_1+m_2+2)-tuple of elements of $X\times Y$ is non-connected. Let

$$\big((x_0,y_0),(x_1,y_1),\ldots,(x_{m_1+m_2+1},y_{m_1+m_2+1})\big)$$

be an arbitrary connected (m_1+m_2+2)-tuple of distinct elements of $X\times Y$. We prove that there exist only the following two cases:

Case 1. There exist m_1+2 distinct elements

$$i_0,i_1,\ldots,i_{m_1+1}\in\{0,\ldots,m_1+m_2+2\}$$

such that $i_0<i_1<\cdots<i_{m_1+1}$ and $x_{i_0}\neq x_{i_1}\neq\cdots\neq x_{i_{m_1+1}}$.

In this case there exists a connected (m_1+2)-tuple of distinct elements of X. This contradicts the equality $h(X)=m_1$.

Case 2. There exist m_2+2 distinct elements

$$i_0,i_1,\ldots,i_{m_2+1}\in\{0,\ldots,m_1+m_2+2\}$$

such that $i_0<i_1<\cdots<i_{m_2+1}$ and $y_{i_0}\neq y_{i_1}\neq\cdots\neq y_{i_{m_2+1}}$.

In this case there exists a connected (m_2+2)-tuple of distinct elements of Y. This contradicts the equality $h(Y)=m_2$.

Let k be the maximum element of $\{0,1,\ldots,m_1+m_2+1\}$ for which there exist $k+1$ distinct elements

$$i_0,i_1,\ldots,i_k\in\{0,\ldots,m_1+m_2+2\}$$

such that $i_0<i_1<\cdots<i_k$ and $x_{i_0}\neq x_{i_1}\neq\cdots\neq x_{i_k}$. We prove that for every $p\in\{0,\ldots,k-1\}$ and $i_p<i<i_{p+1}$ we have $x_i\in\{x_{i_p},x_{i_{p+1}}\}$. Indeed, by the definition of k we have $x_i\in\{x_{i_0},x_{i_1},\ldots,x_{i_k}\}$. If $x_i\in\{x_{i_0},\ldots,x_{i_{p-1}}\}$, then $x_{i_p}\in\mathbf{U}_{x_i}$ and $x_i\in\mathbf{U}_{x_{i_p}}$ which contradicts the fact that the space X is a T_0-space. Therefore, $x_i\notin\{x_{i_0},\ldots,x_{i_{p-1}}\}$. Similar we can prove that $x_i\notin\{x_{i_{p+2}},\ldots,x_{m_1+m_2+1}\}$. Since the space Y is a T_0-space and the pairs $(x_{i_p},y_{i_p}),\ldots,(x_{i_{p+1}},y_{i_{p+1}})$ are distinct, at least $i_{p+1}-i_p$ elements of $y_{i_p},\ldots,y_{i_{p+1}}$ are distinct. Similar we can prove that the elements $y_0,\ldots,y_{i_0}$ and the elements $y_{i_k},\ldots,y_{m_1+m_2+1}$ are distinct. By the above, there exist

$$i_0+1+i_1-(i_0+1)+i_2-(i_1+1)+\cdots+m_1+m_2+1-i_k=m_1+m_2+2-k$$

distinct elements of $\{y_0, \dots, y_{m_1+m_2+1}\}$. Therefore, if $k < m_1 + 1$, then there exist at least $m_2 + 2$ distinct elements of $\{y_0, \dots, y_{m_1+m_2+1}\}$.

Thus, $h(X \times Y) = m_1 + m_2$. $\square$

Proposition 2.14 *(Product Theorem for h). Let X_λ, $\lambda \in \Lambda$, be non-empty Alexandroff T_0-spaces. Then,*

$$h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \max\Big\{\sum\{h(X_\lambda)\colon\ \lambda \in \Lambda'\}\colon\ \Lambda' \in \Lambda_{\mathrm{fin}}\Big\},$$

where Λ_{fin} is the family of all finite subsets of Λ, if max *exists, and*

$$h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \infty$$

otherwise.

Proof. Let $\Lambda' \in \Lambda_{\mathrm{fin}}$. We prove that

$$\sum\{h(X_\lambda)\colon\ \lambda \in \Lambda'\} \leqslant h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big).$$

By Proposition 2.13,

$$h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}\Big) = \sum\{h(X_\lambda)\colon\ \lambda \in \Lambda'\}.$$

So, it suffices to prove that

$$h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}\Big) \leqslant h\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big).$$

Let $h(\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}) = k$, where $k \in \{0, 1, \dots\}$. Then, there exists a connected $(k+1)$-tuple $(\{x^0_\lambda\}_{\lambda\in\Lambda'}, \dots, \{x^k_\lambda\}_{\lambda\in\Lambda'})$ of distinct elements of the product $\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}$. We choose a point $x^*_\lambda \in X_\lambda$ for every $\lambda \in \Lambda \setminus \Lambda'$ and consider the points $\{{}^*x^i_\lambda\}_{\lambda\in\Lambda}$, $i = 0, \dots, k$ of $\prod\{X_\lambda\colon\ \lambda \in \Lambda\}$ such that

$${}^*x^i_\lambda = \begin{cases} x^i_\lambda, & \text{if } \lambda \in \Lambda', \\ x^*_\lambda, & \text{if } \lambda \notin \Lambda'. \end{cases}$$

Then, $(\{{}^*x^0_\lambda\}_{\lambda\in\Lambda}, \dots, \{{}^*x^k_\lambda\}_{\lambda\in\Lambda})$ is a connected $(k+1)$-tuple of distinct elements of $\prod\{X_\lambda\colon\ \lambda \in \Lambda\}$. Therefore, $h(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}) \geqslant k$.

Conversely, let $h(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}) = k$, where $k \in \{0, 1, \dots\}$. Then, there exists a connected $(k+1)$-tuple $(\{{}^*x^0_\lambda\}_{\lambda\in\Lambda}, \dots, \{{}^*x^k_\lambda\}_{\lambda\in\Lambda})$ of distinct elements of $\prod\{X_\lambda\colon\ \lambda \in \Lambda\}$. Since k is a positive integer, there exist $\Lambda' \in \Lambda_{\mathrm{fin}}$ and distinct points $\{x^0_\lambda\}_{\lambda\in\Lambda'}, \dots, \{x^k_\lambda\}_{\lambda\in\Lambda'}$ of $\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}$ such that $x^i_\lambda = {}^*x^i_\lambda$ for every $\lambda \in \Lambda'$. Therefore, $(\{x^0_\lambda\}_{\lambda\in\Lambda'}, \dots, \{x^k_\lambda\}_{\lambda\in\Lambda'})$ is a connected $(k+1)$-tuple of distinct elements of $\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}$. This means that $h(\prod\{X_\lambda\colon\ \lambda \in \Lambda'\}) \geqslant k$ and, therefore, $\max\{\sum\{h(X_\lambda)\colon\ \lambda \in \Lambda'\}\colon\ \Lambda' \in \Lambda_{\mathrm{fin}}\} \geqslant h(\prod\{X_\lambda\colon\ \lambda \in \Lambda\})$. $\square$

Proposition 2.15. *Let $k \in \{0, 1, \dots\}$, X be an Alexandroff T_0-space and $x \in X$. Then, $h_x(X) \leqslant k$ if and only if $h(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant k - 1$.*

Proof. Let $h_x(X) \leqslant k$. We prove that $h(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant k-1$. Consider a connected $(t+1)$-tuple $(\alpha_0, \dots, \alpha_t)$ of distinct elements of $\mathrm{Bd}_X(\mathbf{U}_x)$. We prove that $t < k$. Suppose that $t \geqslant k$. Since $\alpha_t \in \mathrm{Bd}_X(\mathbf{U}_x)$, we have $\mathbf{U}_{\alpha_t} \cap \mathbf{U}_x \neq \emptyset$. Let $\alpha_{t+1} \in \mathbf{U}_{\alpha_t} \cap \mathbf{U}_x$. Then, $\alpha_t \in \mathrm{Cl}_X(\{\alpha_{t+1}\})$. Also, since $\{\alpha_0, \dots, \alpha_t\} \subseteq \mathrm{Bd}_X(\mathbf{U}_x)$ and $\alpha_{t+1} \in \mathbf{U}_x$, we have $\alpha_{t+1} \notin \{\alpha_0, \dots, \alpha_t\}$. Therefore, the $(t+2)$-tuple $(\alpha_0, \dots, \alpha_t, \alpha_{t+1})$ of distinct elements

of X is connected and $\{\alpha_0, \ldots, \alpha_t, \alpha_{t+1}\} \cap \mathbf{U}_x = \{\alpha_{t+1}\}$ which is a contradiction with $h_x(X) \leqslant k$. Thus, $h(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant k-1$.

Conversely, let $h(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant k-1$. We prove that $h_x(X) \leqslant k$. Consider a connected $(t+1)$-tuple $(\alpha_0, \ldots, \alpha_t)$ of distinct elements of X with $\{\alpha_0, \ldots, \alpha_t\} \cap \mathbf{U}_x = \{\alpha_t\}$. We prove that $t \leqslant k$. First we show that $\alpha_i \in \mathrm{Bd}_X(\mathbf{U}_x)$ for every $i \in \{0, \ldots, t-1\}$. Indeed, let $i \in \{0, \ldots, t-1\}$. Then, $\alpha_i \in \mathrm{Cl}_X(\mathbf{U}_{\alpha_t}) \subseteq \mathrm{Cl}_X(\mathbf{U}_x)$. Also, $\alpha_i \notin \mathbf{U}_x$. Thus, $\alpha_i \in \mathrm{Bd}_X(\mathbf{U}_x)$. By the above the t-tuple $(\alpha_0, \ldots, \alpha_{t-1})$ of distinct elements of $\mathrm{Bd}_X(\mathbf{U}_x)$ is connected. Hence, by assumption, $t-1 \leqslant k-1$ or $t \leqslant k$. Thus, $h_x(X) \leqslant k$. □

3. Small inductive dimension of Alexandroff spaces

The *small inductive dimension of a topological space* X (see for example [4] and [8]), denoted by $\mathrm{ind}(X)$, is defined as follows:

(i) $\mathrm{ind}(X) = -1$ if and only if $X = \emptyset$.
(ii) $\mathrm{ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every point $x \in X$ and each open set U of X with $x \in U$ there exists an open set V of X such that

$$x \in V \subseteq U \quad \text{and} \quad \mathrm{ind}\big(\mathrm{Bd}_X(V)\big) \leqslant k-1.$$

(iii) $\mathrm{ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\mathrm{ind}(X) \leqslant k$ and $\mathrm{ind}(X) > k-1$.
(iv) $\mathrm{ind}(X) = \infty$ if $\mathrm{ind}(X) > k$ for $k \in \{-1, 0, 1, \ldots\}$.

The following fact is evident.

Fact 3.1. Let X be an Alexandroff space. Then, $\mathrm{ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if and only if for every $x \in X$ we have $\mathrm{ind}(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant k-1$.

From the definitions of ind and h we have result which is given in [9].

Proposition 3.2. *For every Alexandroff* T_0*-space* X *we have* $\mathrm{ind}(X) = h(X)$.

Corollary 3.3. *Let* X *be an Alexandroff* T_0*-space. Then,* $\mathrm{ind}(X) = k$*, where* $k \in \{0, 1, \ldots\}$*, if and only if there exists a connected* $(k+1)$*-tuple* $(\alpha_0, \ldots, \alpha_k)$ *of distinct elements of* X *and any* $(k+2)$*-tuple of distinct elements of* X *is non-connected.*

Corollary 3.4. *Let* X *be a finite* T_0*-space of* n *elements. If* $n > 1$*, then* $\mathrm{ind}(X) \leqslant n-1$.

Corollary 3.5. *([2]) Let* X *be an Alexandroff* T_0*-space. Then,* $\mathrm{ind}(X) = 0$ *if and only if* X *is discrete.*

Remark 3.6. We observe that Proposition 3.2 is not true in the case where X is not a T_0-space. Indeed, let $X = \{x_1, x_2, x_3, x_4, x_5\}$ with the topology $\tau = \{\emptyset, \{x_2\}, \{x_1, x_2, x_3\}, \{x_2, x_4, x_5\}, X\}$. We observe that X is not a T_0-space, $\mathrm{ind}(X) = 1$, and $h(X) = 2$.

Also, this example shows that Lemma 1 of [2] is not true if X is not a T_0-space.

The *small inductive dimension of a topological space* X *at the point* $x \in X$ (see for example in case of regular spaces [4]), denoted by $\mathrm{ind}_x(X)$, is defined as follows:

(i) $\mathrm{ind}_x(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every open neighborhood $U \subseteq X$ of the point x there exists an open set V of X such that

$$x \in V \subseteq U \quad \text{and} \quad \mathrm{ind}\big(\mathrm{Bd}_X(V)\big) \leqslant k-1.$$

(ii) $\operatorname{ind}_x(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\operatorname{ind}_x(X) \leqslant k$ and $\operatorname{ind}_x(X) > k - 1$.
(iii) $\operatorname{ind}_x(X) = \infty$ if $\operatorname{ind}_x(X) > k$ for $k \in \{-1, 0, 1, \ldots\}$.
The following fact is evident.

Fact 3.7. Let X be an Alexandroff space and $x \in X$. Then, $\operatorname{ind}_x(X) = k$, where $k \in \{0, 1, \ldots\}$, if and only if we have $\operatorname{ind}(\operatorname{Bd}_X(\mathbf{U}_x)) = k - 1$.

Proposition 3.8. *For every Alexandroff* T_0*-space* X *and for any* $x \in X$ *we have* $\operatorname{ind}_x(X) = h_x(X)$.

Proof. First we prove the inequality $\operatorname{ind}_x(X) \leqslant h_x(X)$. If $h_x(X) = \infty$, then the inequality is true. Let $h_x(X) = k$, where $k \in \{0, 1, \ldots\}$. By Proposition 2.15 we have $h(\operatorname{Bd}_X(\mathbf{U}_x)) \leqslant k - 1$. Hence, by Proposition 3.2, $\operatorname{ind}(\operatorname{Bd}_X(\mathbf{U}_x)) \leqslant k - 1$.

Now, we prove the inequality $h_x(X) \leqslant \operatorname{ind}_x(X)$. If $\operatorname{ind}_x(X) = \infty$, then the inequality is true. Let $\operatorname{ind}_x(X) = k$, where $k \in \{0, 1, \ldots\}$. Then, $\operatorname{ind}(\operatorname{Bd}_X(\mathbf{U}_x)) = k-1$. By Proposition 3.2 we have $h(\operatorname{Bd}_X(\mathbf{U}_x)) \leqslant k - 1$. Therefore, by Proposition 2.15, $h_x(X) \leqslant k$. □

Proposition 3.9. *Let* X *be an Alexandroff* T_0*-space with* $\operatorname{ind}(X) < \infty$*. Then, there exists* $x_0 \in X$ *with* $\{x_0\}$ *open subset of* X *such that* $\operatorname{ind}(X) = \operatorname{ind}_{x_0}(X)$.

Proof. The proof of this proposition follows straight by Propositions 2.8, 3.2 and 3.8. □

Remark 3.10. We observe that Proposition 3.9 is not necessarily true in the case where X is not a T_0-space. Indeed, let $X = \{x_1, x_2, x_3, x_4\}$ with the topology

$$\tau = \big\{\emptyset, \{x_1\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, X\big\}.$$

We observe that X is not a T_0-space, $\operatorname{ind}(X) = 1$, and $\operatorname{ind}_{x_1}(X) = 0$.

Example 3.11. Let $k \in \{0, 1, \ldots\}$. Then, there exists an Alexandroff T_0-space X_k such that $\operatorname{ind}(X_k) = k$.

Indeed, we consider the Alexandroff T_0-space (X, τ_k), where $X = \{0, 1, 2, \ldots\}$ and τ_k is the topology generated by the family

$$\beta = \{\emptyset\} \cup \big\{\{0\}, \{0, 1\}, \ldots, \{0, 1, \ldots, k\}\big\} \cup \big\{\{i\}\colon\ i = k + 1, k + 2, \ldots\big\},$$

that is the family β is a base for the topology τ_k. Then,

$$\mathbf{U}_i = \begin{cases} \{0, 1, 2, \ldots, i\}, & \text{if } i \in \{0, 1, 2, \ldots, k\}, \\ \{i\}, & \text{if } i \notin \{0, 1, 2, \ldots, k\}. \end{cases}$$

We observe that the $(k+1)$-tuple $(k, \ldots, 2, 1, 0)$ of distinct elements of X is connected. Also, by the definition of the topology τ_k, any $(k + 2)$-tuple of distinct elements of X is non-connected. Thus, $\operatorname{ind}(X) = k$.

4. Addition, decomposition, structure, sum and product theorems for Alexandroff T_0-spaces

Using Corollary 3.3 we can prove easily, without induction, theorems concerning the small inductive dimension for Alexandroff T_0-spaces.

Proposition 4.1 *(Addition Theorem). Let* X *be an Alexandroff* T_0*-space and* X_1*,* X_2 *two subspaces of* X*. Then,* $\operatorname{ind}(X_1 \cup X_2) \leqslant \operatorname{ind}(X_1) + \operatorname{ind}(X_2) + 1$.

Proof. The proof of this proposition follows straight by Propositions 2.11 and 3.2. □

Remark 4.2. (1) We note that the inequality in Proposition 4.1 cannot be replaced by equality. Indeed, let $X = \{a, b, c, d\}$ with the topology

$$\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\big\},$$

$X_1 = \{a\}$, and $X_2 = \{b\}$. We observe that (X, τ) is a finite T_0-space. Moreover,

$$\operatorname{ind}(X_1) = \operatorname{ind}(X_2) = \operatorname{ind}(X_1 \cup X_2) = 0.$$

Therefore, $\operatorname{ind}(X_1 \cup X_2) = 0 < 1 = \operatorname{ind}(X_1) + \operatorname{ind}(X_2) + 1$.

(2) We note that the inequality

$$\operatorname{ind}(X_1 \cup X_2) \leqslant \operatorname{ind}(X_1) + \operatorname{ind}(X_2) + 1$$

in Proposition 4.1 cannot be replaced by the inequality

$$\operatorname{ind}(X_1 \cup X_2) \leqslant \operatorname{ind}(X_1) + \operatorname{ind}(X_2).$$

Indeed, let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{b\}, \{a, b\}\}$, $X_1 = \{a\}$, and $X_2 = \{b, c\}$. We observe that (X, τ) is a finite T_0-space,

$$\operatorname{ind}(X_1) = 0, \qquad \operatorname{ind}(X_2) = 1, \quad \text{and} \quad \operatorname{ind}(X_1 \cup X_2) = 2.$$

Therefore, $\operatorname{ind}(X_1 \cup X_2) = 2 > 1 = \operatorname{ind}(X_1) + \operatorname{ind}(X_2)$.

Proposition 4.3 *(First Decomposition Theorem). Let X be an Alexandroff T_0-space. If $\operatorname{ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, then X is union of two disjoint sets X_0 and X_1 with X_0 discrete in itself such that X_0 is open and dense in X, $\operatorname{ind}(X_0) = 0$, and $\operatorname{ind}(X_1) = k - 1$.*

Proof. Let $\operatorname{ind}(X) = k$, where $k \in \{0, 1, \ldots\}$. By Proposition 3.2 we have $h(X) = k$. We consider the sets $X_0 = \{x \in X\colon \{x\} \text{ is open}\}$ and $X_1 = X \setminus X_0$. By Proposition 2.7, X_0 is dense X. Also, it is clear that X_0 is open in X and discrete in itself. We observe that $h(X_0) = 0$ and, therefore, $\operatorname{ind}(X_0) = 0$. We prove that $\operatorname{ind}(X_1) = k - 1$ or equivalently $h(X_1) = k - 1$. Since $h(X) = k$, there exists a connected $(k + 1)$-tuple $(\alpha_0, \ldots, \alpha_k)$ of distinct elements of X. We observe that $\alpha_0, \ldots, \alpha_{k-1} \in X_1$. Therefore, the k-tuple $(\alpha_0, \ldots, \alpha_{k-1})$ of distinct elements of X_1 is connected in the subspace X_1. It suffices to prove that any $(k + 1)$-tuple of distinct elements of X_1 is non-connected in the subspace X_1. We suppose that there exists a connected $(k + 1)$-tuple $(\beta_0, \ldots, \beta_k)$ of distinct elements of X_1 in the subspace X_1. Since $\beta_k \notin X_0$, $\mathbf{U}_{\beta_k} \neq \{\beta_k\}$ and, therefore, there exists $\beta_{k+1} \in X_0$ such that $\beta_{k+1} \subseteq \mathbf{U}_{\beta_k}$. Hence the $(k+2)$-tuple $(\beta_0, \ldots, \beta_k, \beta_{k+1})$ of distinct elements of X is connected. Since $h(X) = k$, this is a contradiction. Thus, $h(X_1) = k - 1$ or equivalently $\operatorname{ind}(X_1) = k - 1$. □

By Propositions 4.1 and 4.3 we have the following proposition.

Proposition 4.4 *(Second Decomposition Theorem). Let X be an Alexandroff T_0-space. Then, $\operatorname{ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if and only if X is a union of $k + 1$ discrete sets X_i, $i = 0, \ldots, k$.*

The following proposition characterizes dimension ind in the class of all Alexandroff T_0-spaces.

Proposition 4.5 *(Structure Theorem). Let X be an Alexandroff T_0-space. Then,* $\operatorname{ind}(X) = k$, *where* $k \in \{0, 1, \ldots\}$, *if and only if X is a union of $k+1$ disjoint sets X_i, discrete in themselves, $i = 0, \ldots, k$, such that X_i is open and dense in $X_i \cup \cdots \cup X_k$, $i = 0, \ldots, k$.*

Proof. Applying Proposition 4.3 we have:

X is union of two disjoint sets X_0 and X_1' with X_0 discrete in itself such that X_0 is open and dense in X, $\operatorname{ind}(X_0) = 0$, and $\operatorname{ind}(X_1') = k - 1$.
X_1' is a union of two disjoint sets X_1 and X_2' with X_1 discrete in itself such that X_1 is open and dense in X_1', $\operatorname{ind}(X_1) = 0$, and $\operatorname{ind}(X_2') = (k-1) - 1 = k - 2$.
X_2' is a union of two disjoint sets X_2 and X_3' with X_2 discrete in itself such that X_2 is open and dense in X_2', $\operatorname{ind}(X_2) = 0$, and $\operatorname{ind}(X_3') = (k-2) - 1 = k - 3$.

Continuing in the same manner we have:

X_{k-1}' is a union of two disjoint sets X_{k-1} and X_k with X_{k-1} discrete in itself such that X_{k-1} is open and dense in X_{k-1}', $\operatorname{ind}(X_{k-1}) = 0$, and $\operatorname{ind}(X_k) = k - k = 0$.

Therefore, we have constructed subsets $X_0, X_1, \ldots, X_k$ of X which have the following properties:

(1) $X = X_0 \cup X_1 \cup \cdots \cup X_k$.
(2) $X_i \cap X_j = \emptyset$ for every $i, j \in \{0, 1, \ldots, k\}$ with $i \neq j$.
(3) $\operatorname{ind}(X_i \cup \cdots \cup X_k) = \operatorname{ind}(X \setminus (X_0 \cup \cdots \cup X_{i-1})) = \operatorname{ind}(X_i') = k - i$, $i = 1, \ldots, k$.
(4) X_0 is open and dense in $X_0 \cup \cdots \cup X_k = X$ and X_i is open and dense in $X_i \cup \cdots \cup X_k = X \setminus (X_0 \cup \cdots \cup X_{i-1}) = X_i'$, $i = 1, \ldots, k$.

Conversely, let us prove by induction that for $i = 0, 1, \ldots, k$,

$$\operatorname{ind}(X_i \cup \cdots \cup X_k) = k - i.$$

For $i = k$, $\operatorname{ind}(X_k) = 0$. Suppose that $\operatorname{ind}(X_i \cup \cdots \cup X_k) = k - i$, where $i = 1, \ldots, k$. We prove that $\operatorname{ind}(X_{i-1} \cup X_i \cup \cdots \cup X_k) = k - i + 1$. By Proposition 4.1,

$$\begin{aligned} \operatorname{ind}(X_{i-1} \cup X_i \cup \cdots \cup X_k) &\leqslant \operatorname{ind}(X_{i-1}) + \operatorname{ind}(X_i \cup \cdots \cup X_k) + 1 \\ &= 0 + (k - i) + 1 = k - i + 1. \end{aligned}$$

By Corollary 3.3 there exists a connected $(k-i+1)$-tuple $(\alpha_0, \ldots, \alpha_{k-i+1})$ of distinct elements of the union $X_i \cup \cdots \cup X_k$. Since X_{i-1} is dense in the union $X_{i-1} \cup X_i \cup \cdots \cup X_k$, there is $\alpha_{k-i+2} \in X_{i-1}$ such that $\mathbf{U}_{\alpha_{k-i+2}} \subseteq \mathbf{U}_{\alpha_{k-i+1}}$. Hence $(\alpha_0, \ldots, \alpha_{k-i+1}, \alpha_{k-i+2})$ is a connected $(k-i+2)$-tuple of distinct elements of $X_{i-1} \cup X_i \cup \cdots \cup X_k$. Thus, $\operatorname{ind}(X_{i-1} \cup X_i \cup \cdots \cup X_k) = k - i + 1$. Particularly, $\operatorname{ind}(X) = \operatorname{ind}(X_0 \cup \cdots \cup X_k) = k$. □

Corollary 4.6. *Let X be an Alexandroff T_0-space with* $\operatorname{ind}(X) = k$, *where* $k \in \{0, 1, \ldots\}$. *Then, for the decomposition from Proposition 4.5, we have*

$$\operatorname{ind}(X_i \cup \cdots \cup X_k) = k - i, \quad i = 0, \ldots, k.$$

The Structure Theorem allows to sharpen results from [3].

Proposition 4.7 *(Finite Sum Theorem for ind). Let X be an Alexandroff T_0-space and X_1, X_2 are closed subspaces of X with $X = X_1 \cup X_2$. Then,* $\operatorname{ind}(X) = \max\{\operatorname{ind}(X_1), \operatorname{ind}(X_2)\}$. *Moreover, if $X_1 = \bigcup_{i=0}^{m} X_i^1$ and $X_2 = \bigcup_{j=0}^{k} X_j^2$ are decompositions of X_1 and X_2 due to the Structure Theorem, then $X = \bigcup_{s=0}^{\max\{m,k\}} X_s^*$, where*

$$X_s^* = (X_s^1 \setminus X_2) \cup \left(X_s^1 \cap \left(\bigcup_{i=0}^{s} X_i^2 \right) \right) \cup \left(X_s^2 \cap \left(\bigcup_{i=0}^{s} X_i^1 \right) \right) \cup (X_s^2 \setminus X_1),$$

is the decomposition of X.

Proof. By Propositions 2.12 and 3.2 we have $\operatorname{ind}(X) = \max\{\operatorname{ind}(X_1), \operatorname{ind}(X_2)\}$. We prove the second part of proposition. By assumption we observe that $X = \bigcup_{s=0}^{\max\{m,k\}} X_s^*$ and $X_s^* \cap X_t^* = \emptyset$ for $s \neq t$.

Let $s \in \{0, \ldots, \max\{m,k\}\}$. First we prove that the space X_s^* is discrete. Let $x \in X_s^*$. It suffices to prove that $\mathbf{U}_x \cap X_s^* = \{x\}$. We consider the following cases:

(1) Let $x \in X_s^1 \setminus X_2$. Since X_s^1 is discrete, $\mathbf{U}_x \cap X_s^1 = \{x\}$. Moreover, since X_2 is closed subset of X and $x \in X \setminus X_2$, we have $\mathbf{U}_x \subseteq X \setminus X_2$. Therefore, $\mathbf{U}_x \cap X_s^* = \mathbf{U}_x \cap (X_s^1 \setminus X_2) = \{x\}$.

(2) The case $x \in X_s^2 \setminus X_1$ is examined analogously.

(3) Let $x \in X_s^1 \cap (\bigcup_{i=0}^{s} X_i^2)$. We prove that $\mathbf{U}_x \cap X_s^* \subseteq \mathbf{U}_x \cap (X_s^1 \cup X_s^2)$. Indeed, let $a \in \mathbf{U}_x \cap X_s^*$. Then, $a \in \mathbf{U}_x$. Also, by the definition of X_s^*, $a \in X_s^1$ or $a \in X_s^2$ which means that $a \in \mathbf{U}_x \cap (X_s^1 \cup X_s^2)$. Hence,

$$\mathbf{U}_x \cap X_s^* \subseteq \mathbf{U}_x \cap \left(X_s^1 \cup X_s^2\right) = \left(\mathbf{U}_x \cap X_s^1\right) \cup \left(\mathbf{U}_x \cap X_s^2\right). \tag{4.1}$$

Since $x \in X_s^1$ and X_s^1 is discrete, $\mathbf{U}_x \cap X_s^1 = \{x\}$. We prove that either $\mathbf{U}_x \cap X_s^2 = \{x\}$ or $\mathbf{U}_x \cap X_s^2 = \emptyset$. Indeed, since $x \in \bigcup_{i=0}^{s} X_i^2$, either $x \in X_s^2$ or $x \in \bigcup_{i=0}^{s-1} X_i^2$. If $x \in X_s^2$, then $\mathbf{U}_x \cap X_s^2 = \{x\}$ because X_s^2 is discrete. If $x \in \bigcup_{i=0}^{s-1} X_i^2$, then there exists $i_0 \in \{0, \ldots, s-1\}$ such that $x \in X_{i_0}^2$. Since $X_{i_0}^2$ is open in $\bigcup_{i=i_0}^{k} X_i^2$, we have $\mathbf{U}_x \cap \bigcup_{i=i_0}^{k} X_i^2 \subseteq X_{i_0}^2$. Also, since $X_{i_0}^2 \cap X_s^2 = \emptyset$, we have $\mathbf{U}_x \cap X_s^2 = \emptyset$. Thus, by relation (4.1) we have $\mathbf{U}_x \cap X_s^* = \{x\}$.

(4) The case $x \in X_s^2 \cap (\bigcup_{i=0}^{s} X_i^1)$ is examined analogously.

Now, we prove that the space X_s^* is open in $\bigcup_{i=s}^{\max\{m,k\}} X_i^*$. Let $x \in X_s^*$. It suffices to prove that $\mathbf{U}_x \cap \bigcup_{i=s}^{\max\{m,k\}} X_i^* \subseteq X_s^*$ or $\mathbf{U}_x \cap X_t^* = \emptyset$ for $t > s$. We consider the following cases:

(1′) Let $x \in X_s^1 \setminus X_2$. Since X_s^1 is open in $\bigcup_{i=s}^{m} X_i^1$, we have $\mathbf{U}_x \cap \bigcup_{i=s}^{m} X_i^1 \subseteq X_s^1$ and, consequently, $\mathbf{U}_x \cap X_t^1 = \emptyset$ for $t > s$. Moreover, since X_2 is closed subset of X and $x \in X \setminus X_2$, we have $\mathbf{U}_x \subseteq X \setminus X_2$. We prove that $\mathbf{U}_x \cap X_t^* = \emptyset$ for $t > s$. Indeed, let $t > s$ and $a \in \mathbf{U}_x \cap X_t^*$. Since $\mathbf{U}_x \subseteq X \setminus X_2$, $a \in X_t^1 \setminus X_2$ which is a contradiction with $\mathbf{U}_x \cap X_t^1 = \emptyset$. Hence, $\mathbf{U}_x \cap X_t^* = \emptyset$ for $t > s$.

(2′) The case $x \in X_s^2 \setminus X_1$ is examined analogously.

(3′) Let $x \in X_s^1 \cap (\bigcup_{i=0}^{s} X_i^2)$. Then, $x \in X_s^1$. Since X_s^1 is open in $\bigcup_{i=s}^{m} X_i^1$, we have $\mathbf{U}_x \cap \bigcup_{i=s}^{m} X_i^1 \subseteq X_s^1$ and, consequently,

$$\mathbf{U}_x \cap X_t^1 = \emptyset \quad \text{for } t > s. \tag{4.2}$$

Moreover, $\mathbf{U}_x \cap X_t^2 = \emptyset$ for $t > s$. Indeed, since $x \in \bigcup_{i=0}^{s} X_i^2$, there exists $i_0 \in \{0, \ldots, s\}$ such that $x \in X_{i_0}^2$. Since $X_{i_0}^2$ is open in $\bigcup_{i=i_0}^{k} X_i^2$, we have $\mathbf{U}_x \cap X_t^2 = \emptyset$ for $t > i_0$ and, consequently,

$$\mathbf{U}_x \cap X_t^2 = \emptyset \quad \text{for } t > s. \tag{4.3}$$

Therefore, by the relations (4.2) and (4.3) and the definition of X_s^*, we have $\mathbf{U}_x \cap X_t^* = \emptyset$ for $t > s$.

(4′) The case $x \in X_s^2 \cap (\bigcup_{i=0}^{s} X_i^1)$ is examined analogously.

Finally, we prove that the space X_s is dense in $\bigcup_{i=s}^{\max\{m,k\}} X_i^*$. Let O be a non-empty open subset of $Z = \bigcup_{i=s}^{\max\{m,k\}} X_i^*$. It suffices to prove that $O \cap X_s^* \neq \emptyset$. Without loss of generality we can consider that $O = \mathbf{U}_x \cap Z$ for some $\mathbf{U}_x$, $x \in Z$. We consider the following three cases:

$(1'')$ Let $O \cap (X_1 \setminus X_2) \neq \emptyset$. Since $O \subseteq Z$, there exists $i_0 \in \{s, \ldots, m\}$ such that $O \cap (X_{i_0}^1 \setminus X_2) \neq \emptyset$. This means that $O \cap \bigcup_{i=s}^m X_i^1$ is a non-empty open subset of $\bigcup_{i=s}^m X_i^1$. Since X_s^1 is dense in $\bigcup_{i=s}^m X_i^1$, we have $O \cap X_s^1 \neq \emptyset$. Therefore, $O \cap (X_s^1 \setminus X_2) \neq \emptyset$. By the definition of X_s^* we have $O \cap X_s^* \neq \emptyset$.

$(2'')$ The case $O \cap (X_2 \setminus X_1) \neq \emptyset$ is examined analogously.

$(3'')$ Let $O \cap (X_1 \setminus X_2) = \emptyset$ and $O \cap (X_2 \setminus X_1) = \emptyset$. Then, $O \subseteq X_1 \cap X_2$. Moreover, since $x \in O$, we have

$$x \in \bigcup_{l=s}^{\max\{m,k\}} \left[\left(X_l^1 \cap \left(\bigcup_{i=0}^{l} X_i^2 \right) \right) \cup \left(X_l^2 \cap \left(\bigcup_{i=0}^{l} X_i^1 \right) \right) \right].$$

Let $x \in X_l^1$ for some $l \geqslant s$ (the case $x \in X_l^2$ for some $l \geqslant s$, is examined analogously). Since $x \in X_l^1$ and X_s^1 is dense in $\bigcup_{i=s}^m X_i^1$, we have $O \cap X_s^1 \neq \emptyset$. Let $y \in O \cap X_s^1$. Then, $\mathbf{U}_y \cap Z \subseteq O \subseteq X_1 \cap X_2$. Since $y \in X_2$, there exists $i_0 \in \{0, \ldots, k\}$ such that $y \in X_{i_0}^2$. We consider the following two cases for the element i_0:

Case 1. $i_0 \leqslant s$.

In this case $y \in X_s^1 \cap X_{i_0}^2 \subseteq X_s^1 \cap (\bigcup_{i=0}^s X_i^2) \subseteq X_s^*$. Hence, $y \in O \cap X_s^* \neq \emptyset$.

Case 2. $i_0 > s$.

In this case since X_s^2 is dense in $\bigcup_{i=s}^k X_i^2$ and $y \in X_{i_0}^2 \subseteq \bigcup_{i=s}^k X_i^2$, we have $\mathbf{U}_y \cap X_s^2 \neq \emptyset$. Let $z \in \mathbf{U}_y \cap X_s^2$. Since X_s^1 is open in $\bigcup_{i=s}^m X_i^1$ and $y \in X_s^1$, we have $\mathbf{U}_y \cap \bigcup_{i=s}^m X_i^1 \subseteq X_s^1$ and, consequently, $\mathbf{U}_y \cap X_i^1 = \emptyset$ for $i > s$. Therefore, $z \notin X_i^1$ for $i > s$. But $z \in X_1$. Thus, $z \in \bigcup_{i=0}^s X_i^1$ and, consequently, $z \in X_s^2 \cap (\bigcup_{i=0}^s X_i^1) \subseteq X_s^*$. Also, $z \in \mathbf{U}_y \cap Z \subseteq O$. Hence, $z \in O \cap X_s^* \neq \emptyset$. This means that $O \cap X_s^* \neq \emptyset$.

Applying Proposition 4.5 (Structure Theorem) we obtain the desired result. □

Proposition 4.8 *(Sum Theorem for ind). Let X be an Alexandroff T_0-space and X_λ, $\lambda \in \Lambda$, closed subspaces of X. Then,*

$$\operatorname{ind}\Big(\bigcup\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \sup\big\{\operatorname{ind}(X_\lambda)\colon\ \lambda \in \Lambda\big\}.$$

Proof. The proof of this proposition follows straight by Propositions 2.12 and 3.2. □

Proposition 4.9 *(Finite Product Theorem for ind). For Alexandroff T_0-spaces X and Y we have $\operatorname{ind}(X \times Y) = \operatorname{ind}(X) + \operatorname{ind}(Y)$. Moreover if $\operatorname{ind}(X) = m$ and $\operatorname{ind}(Y) = k$, where $m, k \in \{0, 1, \ldots\}$, and $X = \bigcup_{i=0}^m X_i$, $Y = \bigcup_{j=0}^k Y_i$ are decompositions of X, Y due to the Structure Theorem, then $X \times Y = \bigcup_{l=0}^{m+k} Z_l$, where $Z_l = \bigcup_{s=0}^l (X_s \times Y_{l-s})$, is the decomposition of $X \times Y$.*

Proof. By Propositions 2.13 and 3.2 we have $\operatorname{ind}(X \times Y) = \operatorname{ind}(X) + \operatorname{ind}(Y)$. We prove the second part of proposition. By assumption we observe that $X \times Y = \bigcup_{l=0}^{m+k} Z_l$ and $Z_l \cap Z_t = \emptyset$ for $l \neq t$.

Let $l \in \{0, \ldots, m+k\}$. First we prove that the space Z_l is discrete. Let $(x, y) \in Z_l$. It suffices to prove that $(\mathbf{U}_x \times \mathbf{U}_y) \cap Z_l = \{(x, y)\}$. By the definition of Z_l, there exists $s_0 \in \{0, \ldots, l\}$ such that $(x, y) \in X_{s_0} \times Y_{l-s_0}$. Then, $x \in X_{s_0}$ and $y \in Y_{l-s_0}$.

Since X_{s_0} is discrete, $\mathbf{U}_x \cap X_{s_0} = \{x\}$. Also, since X_{s_0} is open in $\bigcup_{i=s_0}^m X_i$, we have $\mathbf{U}_x \cap \bigcup_{i=s}^m X_i \subseteq X_{s_0}$ and, consequently, $\mathbf{U}_x \cap X_s = \emptyset$ for $s > s_0$. This means that $\mathbf{U}_x \subseteq \bigcup_{i=0}^{s_0} X_i$.

Since Y_{l-s_0} is discrete, $\mathbf{U}_y \cap Y_{l-s_0} = \{y\}$. Also, since Y_{l-s_0} is open in $\bigcup_{j=l-s_0}^k Y_j$, we have $\mathbf{U}_y \cap \bigcup_{j=l-s_0}^k Y_j \subseteq Y_{l-s_0}$ and, consequently, $\mathbf{U}_y \cap Y_s = \emptyset$ for $s > l - s_0$. This means that $\mathbf{U}_y \subseteq \bigcup_{j=0}^{l-s_0} Y_j$.

Let $s > s_0$. Then, $\mathbf{U}_x \cap X_s = \emptyset$. Therefore,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_s \times Y_{l-s}) = (\mathbf{U}_x \cap X_s) \times (\mathbf{U}_y \cap Y_{l-s}) = \emptyset.$$

Now, let $s < s_0$. Then, $l - s > l - s_0$ and, therefore, $\mathbf{U}_y \cap Y_{l-s} = \emptyset$. Hence,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_s \times Y_{l-s}) = (\mathbf{U}_x \cap X_s) \times (\mathbf{U}_y \cap Y_{l-s}) = \emptyset.$$

By the above we have

$$\begin{aligned}(\mathbf{U}_x \times \mathbf{U}_y) \cap Z_l &= (\mathbf{U}_x \times \mathbf{U}_y) \cap \bigcup_{s=0}^{l} (X_s \times Y_{l-s}) = (\mathbf{U}_x \times \mathbf{U}_y) \cap (X_{s_0} \times Y_{l-s_0}) \\ &= (\mathbf{U}_x \cap X_{s_0}) \times (\mathbf{U}_y \cap Y_{l-s_0}) = \{x\} \times \{y\} = \big\{(x,y)\big\}.\end{aligned}$$

Now, we prove that the space Z_l is open in $\bigcup_{i=l}^{k+m} Z_i$. Let $(x,y) \in Z_l$. It suffices to prove that $(\mathbf{U}_x \times \mathbf{U}_y) \cap \bigcup_{i=l}^{k+m} Z_i \subseteq Z_l$ or $(\mathbf{U}_x \times \mathbf{U}_y) \cap Z_t = \emptyset$ for $t > l$. By the definition of Z_l, there exists $s_0 \in \{0, \ldots, l\}$ such that $(x,y) \in X_{s_0} \times Y_{l-s_0}$. Then, $x \in X_{s_0}$ and $y \in Y_{l-s_0}$.

Let $t > l$. Then, $Z_t = \bigcup_{s=0}^{t} (X_s \times Y_{t-s})$. We show that

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap \bigcup_{s=0}^{t} (X_s \times Y_{t-s}) = \emptyset.$$

Since X_{s_0} is open in $\bigcup_{i=s_0}^{m} X_i$, we have $\mathbf{U}_x \cap \bigcup_{i=s_0}^{m} X_i \subseteq X_{s_0}$ and, consequently, $\mathbf{U}_x \cap X_s = \emptyset$ for $s > s_0$. Therefore,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_s \times Y_{t-s}) = (\mathbf{U}_x \cap X_s) \times (\mathbf{U}_y \cap Y_{t-s}) = \emptyset, \quad s > s_0.$$

Moreover, since Y_{t-s_0} is open in $\bigcup_{j=t-s_0}^{k} Y_j$, we have $\mathbf{U}_y \cap \bigcup_{j=t-s_0}^{k} Y_j \subseteq Y_{t-s_0}$ and, consequently, $\mathbf{U}_y \cap Y_s = \emptyset$ for $s > t - s_0$ or $\mathbf{U}_y \cap Y_{t-s} = \emptyset$ for $s < s_0$. Therefore,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_s \times Y_{t-s}) = (\mathbf{U}_x \cap X_s) \times (\mathbf{U}_y \cap Y_{t-s}) = \emptyset, \quad s < s_0.$$

We look the final case $s = s_0$. Since Y_{l-s_0} is open in $\bigcup_{j=l-s_0}^{k} Y_j$, we have $\mathbf{U}_y \cap \bigcup_{j=l-s_0}^{k} Y_j \subseteq Y_{l-s_0}$. Therefore, since $t - s_0 > l - s_0$, we have $\mathbf{U}_y \cap Y_{t-s_0} = \emptyset$. Hence,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_{s_0} \times Y_{t-s_0}) = (\mathbf{U}_x \cap X_{s_0}) \times (\mathbf{U}_y \cap Y_{t-s_0}) = \emptyset.$$

Finally, we prove that the space Z_l is dense in $\bigcup_{i=l}^{k+m} Z_i$. Let O be a non-empty open subset of $Z = \bigcup_{i=l}^{k+m} Z_i$. It suffices to prove that $O \cap Z_l \neq \emptyset$. Without loss of generality we can consider that $O = (\mathbf{U}_x \times \mathbf{U}_y) \cap Z$ for some $(x,y) \in Z$. If $(x,y) \in Z_l$, then we are done. Suppose that $(x,y) \in Z \setminus Z_l$. Then, there exists $t_0 \in \{l+1, \ldots, k+m\}$ such that $(x,y) \in Z_{t_0} = \bigcup_{s=0}^{t_0} (X_s \times Y_{t_0-s})$. Hence, $x \in X_{s_0}$ and $y \in Y_{t_0-s_0}$ for some $t_0 \geqslant l+1$ and $s_0 \leqslant t_0$. We consider the following two cases:

Case 1. $s_0 \leqslant l$.

In this case $x \in \mathbf{U}_x \cap X_{s_0} \neq \emptyset$. Also, since $y \in Y_{t_0-s_0}$ and Y_{l-s_0} is dense in $\bigcup_{i=l-s_0}^{k} Y_i$, we have $\mathbf{U}_y \cap Y_{l-s_0} \neq \emptyset$. Hence,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_{s_0} \times Y_{l-s_0}) = (\mathbf{U}_x \cap X_{s_0}) \times (\mathbf{U}_y \cap Y_{l-s_0}) \neq \emptyset.$$

Since $X_{s_0} \times Y_{l-s_0} \subseteq \bigcup_{s=0}^{l}(X_s \times Y_{l-s}) = Z_l$, we have $O \cap Z_l \neq \emptyset$.

Case 2. $s_0 > l$.

Since $x \in X_{s_0}$ and X_l is dense in $\bigcup_{i=l}^{m} X_i$, we have $\mathbf{U}_x \cap X_l \neq \emptyset$. Now, since $y \in Y_{t_0-s_0}$ and Y_0 is dense in $\bigcup_{i=0}^{m} Y_i$, we have $\mathbf{U}_y \cap Y_0 \neq \emptyset$. Thus,

$$(\mathbf{U}_x \times \mathbf{U}_y) \cap (X_l \times Y_0) = (\mathbf{U}_x \cap X_l) \times (\mathbf{U}_y \cap Y_0) \neq \emptyset.$$

Since $X_l \times Y_0 \subseteq \bigcup_{s=0}^{l}(X_s \times Y_{l-s}) = Z_l$, we have $O \cap Z_l \neq \emptyset$.

Applying Proposition 4.5 (Structure Theorem) we obtain the desired result. □

Proposition 4.10. *(Product Theorem for ind) Let X_λ, $\lambda \in \Lambda$, be non-empty Alexandroff T_0-spaces. Then,*

$$\operatorname{ind}\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \max\Big\{\sum\big\{\operatorname{ind}(X_\lambda)\colon\ \lambda \in \Lambda'\big\}\colon\ \Lambda' \in \Lambda_{\mathrm{fin}}\Big\},$$

where Λ_{fin} is the family of all finite subsets of Λ, if max *exists, and*

$$\operatorname{ind}\Big(\prod\{X_\lambda\colon\ \lambda \in \Lambda\}\Big) = \infty$$

otherwise.

Proof. The proof of this proposition follows straight by Propositions 2.14 and 3.2. □

5. Relations between the small inductive dimension, large inductive dimension and covering dimension in the class of all Alexandroff T_0-spaces

The *large inductive dimension* of a topological space X (see for example [4] and [8]), denoted by $\operatorname{Ind}(X)$, is defined as follows:

(i) $\operatorname{Ind}(X) = -1$ if and only if $X = \emptyset$.
(ii) $\operatorname{Ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if and only if for every pair (F, U) of subsets of X, where F is closed, U is open, and $F \subseteq U$, there exists an open set V of X such that

$$F \subseteq V \subseteq U \quad \text{and} \quad \operatorname{Ind}\big(\operatorname{Bd}_X(V)\big) \leqslant k-1.$$

(iii) $\operatorname{Ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\operatorname{Ind}(X) \leqslant k$ and $\operatorname{Ind}(X) > k-1$.
(iv) $\operatorname{Ind}(X) = \infty$ if $\operatorname{Ind}(X) > k$ for $k \in \{-1, 0, 1, \ldots\}$.

Let X be an Alexandroff topological space. For every closed subset F of X by $\mathbf{U}_F$ we denote the smallest open set containing the set F. That is, $\mathbf{U}_F$ is the intersection of all the open sets containing F.

Proposition 5.1. *Let X be an Alexandroff topological space. Then,* $\operatorname{Ind}(X) \leqslant k$, *where $k \in \{0, 1, \ldots\}$, if and only if for every closed subset F of X we have* $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$.

Proof. Let $\operatorname{Ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, and F be an arbitrary closed subset of X. Then, there exists an open set V of X such that $F \subseteq V \subseteq \mathbf{U}_F$ and $\operatorname{Ind}(\operatorname{Bd}_X(V)) \leqslant k-1$. Since $\mathbf{U}_F$ is the smallest open set containing the set F, we have $V = \mathbf{U}_F$ and, therefore, $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$.

Conversely, we suppose that the space X satisfying the condition of the proposition. We consider a pair (F, U) of subsets of X, where F is closed, U is open, and $F \subseteq U$. Then, $F \subseteq \mathbf{U}_F \subseteq U$ and $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$. Thus, $\operatorname{Ind}(X) \leqslant k$. □

Corollary 5.2. *Let X be an Alexandroff topological space. Then, $\operatorname{Ind}(X) = k$, where $k \in \{0, 1, \ldots\}$ if and only if $\operatorname{Ind}(X) \leqslant k$ and there exists a closed subset F of X such that $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) = k-1$.*

Proposition 5.3. *Let X be an Alexandroff space. Then,*

$$\operatorname{Ind}(X) \leqslant h(X). \tag{5.1}$$

Proof. We prove the relation (5.1) by induction on $h(X) = m$. If $h(X) = \infty$, then (5.1) is true. Let $m = 0$. We prove that $\operatorname{Ind}(X) = 0$. Since $h(X) = 0$, any 2-tuple (α_0, α_1) of distinct elements of X is non-connected. By Corollary 5.2 it suffices to prove that for every closed subset F of X we have $\operatorname{Bd}_X(\mathbf{U}_F) = \emptyset$.

Let F be a closed subset of X. We prove that $\operatorname{Bd}_X(\mathbf{U}_F) = \emptyset$. Suppose that $\operatorname{Bd}_X(\mathbf{U}_F) \neq \emptyset$ and $\beta_0 \in \operatorname{Bd}_X(\mathbf{U}_F) = \operatorname{Cl}_X(\mathbf{U}_F) \setminus \mathbf{U}_F$. Then, $\mathbf{U}_{\beta_0} \cap \mathbf{U}_F \neq \emptyset$. Let $\beta_1 \in \mathbf{U}_{\beta_0} \cap \mathbf{U}_F$. We observe that $\beta_0 \neq \beta_1$. Then, $\beta_0 \in \operatorname{Cl}_X(\{\beta_1\})$ which means that the 2-tuple (β_0, β_1) of distinct elements of X is connected which is a contradiction. Thus, $\operatorname{Bd}_X(\mathbf{U}_F) = \emptyset$.

Suppose that the relation (5.1) is true for all m, $0 \leqslant m \leqslant k-1$, $k \geqslant 1$. We prove the relation (5.1) for $m = k$. Let X be an Alexandroff space with $h(X) = k$. We show that $\operatorname{Ind}(X) \leqslant k$. By Proposition 5.1 it suffices to prove that for every closed subset F of X we have $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$.

Let F be a closed subset of X. We prove that $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$. It suffices to prove that $h(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant k-1$. We consider a connected $(l+1)$-tuple $(\alpha_0, \ldots, \alpha_l)$ of distinct elements of $\operatorname{Bd}_X(\mathbf{U}_F)$ and prove that $l < k$. Suppose that $l \geqslant k$. Since $\alpha_l \in \operatorname{Bd}_X(\mathbf{U}_F) = \operatorname{Cl}_X(\mathbf{U}_F) \setminus \mathbf{U}_F$, we have that $\mathbf{U}_{\alpha_l} \cap \mathbf{U}_F \neq \emptyset$. Let $\alpha_{l+1} \in \mathbf{U}_{\alpha_l} \cap \mathbf{U}_F$. Then, $\alpha_l \in \operatorname{Cl}_X(\{\alpha_{l+1}\})$. Since

$$\{\alpha_0, \ldots, \alpha_l\} \subseteq \operatorname{Bd}_X(\mathbf{U}_F) = \operatorname{Cl}_X(\mathbf{U}_F) \setminus \mathbf{U}_F \quad \text{and} \quad \alpha_{l+1} \in \mathbf{U}_F,$$

we have $\alpha_{l+1} \notin \{\alpha_0, \ldots, \alpha_l\}$. Thus, the $(l+2)$-tuple $(\alpha_0, \ldots, \alpha_l, \alpha_{l+1})$ of distinct elements of X is connected which is a contradiction. □

By Propositions 3.2 and 5.3 we have the following corollary.

Corollary 5.4. *Let X be an Alexandroff T_0-space. Then, $\operatorname{Ind}(X) \leqslant \operatorname{ind}(X)$.*

Define the *order* of a family r of subsets of a space X as follows:

(a) $\operatorname{ord}(r) = -1$ if and only if r consists the empty set only.
(b) $\operatorname{ord}(r) = k$, where $k \in \{0, 1, \ldots\}$, if and only if the intersection of any $k+2$ distinct elements of r is empty and there exist $k+1$ distinct elements of r, whose intersection is not empty.
(c) $\operatorname{ord}(r) = \infty$, if and only if for every $k \in \{0, 1, \ldots\}$ there exist k distinct elements of r, whose intersection is not empty.

The *covering dimension* of a topological space X (see, for example, [4] and [8]), denoted by $\dim(X)$, is defined as follows: $\dim(X) \leqslant k$, where $k \in \{-1, 0, 1, \ldots\}$ if and only if for every finite open cover c of the space X there exists a finite open cover r of X, refinement of c, such that $\operatorname{ord}(r) \leqslant k$.

Proposition 5.5. *([5] and [6]) Let X be an Alexandroff space with a finite open cover of smallest open neighborhoods. Then, $\dim(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if and only if there exists an open cover $\{\mathbf{U}_{x_1}, \ldots, \mathbf{U}_{x_m}\}$ of X such that $\operatorname{ord}(\{\mathbf{U}_{x_1}, \ldots, \mathbf{U}_{x_m}\}) \leqslant k$.*

Example 5.6. Let $X = \{1, 2\}$. We consider on X the following topology $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$. We observe that (X, τ) is a T_0-space. Moreover, $\mathbf{U}_1 = \{1\}$ and $\mathbf{U}_2 = \{1, 2\}$. Therefore, $\operatorname{ind}(X) = 1$ and $\operatorname{Ind}(X) = \dim(X) = 0$.

Example 5.7. Let $X = \{1, 2, 3, 4, 5\}$. We consider on X the topology which has as a basis the family $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$. We observe that (X, τ) is a T_0-space. Moreover, $\mathbf{U}_1 = \{1\}$, $\mathbf{U}_2 = \{1, 2\}$, $\mathbf{U}_3 = \{1, 3\}$, $\mathbf{U}_4 = \{1, 4\}$, $\mathbf{U}_5 = \{1, 5\}$. Therefore, $\operatorname{ind}(X) = \operatorname{Ind}(X) = 1$ and $\dim(X) = 3$.

6. Questions

Question 1. Find a characterization for the large inductive dimension of an Alexandroff T_0-space using connected tuples.

Question 2. Find a characterization for the covering dimension of an Alexandroff T_0-space using connected tuples.

Question 3. What conditions must an Alexandroff T_0-space X satisfy so that $\operatorname{Ind}(X) = \operatorname{ind}(X)$?

Question 4. What conditions must an Alexandroff T_0-space X satisfy so that $\operatorname{ind}(X) = \dim(X)$?

Question 5. What conditions must an Alexandroff T_0-space X satisfy so that $\operatorname{ind}(X) = \dim(X) = \operatorname{Ind}(X)$?

Question 6. Does the following inequality $\operatorname{Ind}(X) \leqslant \dim(X)$ hold for an Alexandroff T_0-space X?

Definition 6.1. A space T is said to be *universal* in a class $\mathbb{P}$ of spaces if:

(1) $T \in \mathbb{P}$ and
(2) for every $X \in \mathbb{P}$ there exists an embedding of X into T.

Question 7. Let $k \in \{0, 1, \ldots\}$. Does there exists a universal space in the class of all Alexandroff T_0-spaces X of weight $\leqslant \tau$, where τ is an infinite cardinal, such that $\operatorname{ind}(X) \leqslant k$?

S.D. Iliadis in the book [7] gave the notion of a saturated class of spaces.

Question 8. Is the class of all Alexandroff T_0-spaces of weight $\leqslant \tau$, where τ is an infinite cardinal, saturated?

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