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On completeness in strong partial b-metric spaces, strong b-metric spaces and the 0-Cauchy completions

Seithuti Moshokoa *, Fanyana Ncongwane

Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria, 0008, South Africa

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ABSTRACT

The purpose of this paper is to introduce a new notion of a strong partial b-metric space, discuss the notions of completeness via variants of Cauchy sequences and provide a 0-Cauchy completion result for the spaces. The class of strong partial b-metric spaces properly lie between the class of strong b-metric spaces and partial b-metric spaces. Finally, we show that the 0-Cauchy completion of a strong partial b-metric space is unique up to isometry. Our completion result coincides with the classical result on the completion of a strong b-metric space. As an application the Banach contraction principle is discussed in this context.

1. Introduction

In [5,6] the notion of a partial metric space was introduced not only as a generalisation of a metric space but due to applicability of partial metric spaces in theoretical computer science. Since then many authors studied partial metric spaces from a vintage point of fixed point theory as well as applications. The paper [7] introduces the notion of a partial b-metric space, as a generalization of partial metric spaces and b-metric spaces. Fixed point theorems are also studied. None or few of the topological properties of partial b-metric spaces are studied in [7]. We can only expect similar problems associated with the continuity of the partial b-metric on X to exists like in b-metric spaces [4]. As a result we can conjecture some problems that exist in studying topological properties in b-metric spaces as highlighted in [4] to be present in partial b-metric spaces. As an alternative to b-metric spaces, strong b-metric spaces were introduced [1] and [4]. These spaces have "good" topological properties as well as nice completion properties [1].

* Corresponding author.
E-mail addresses: MoshokoaSP@tut.ac.za (S. Moshokoa), NcongwaneFT@tut.ac.za (F. Ncongwane).

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In [7] partial b-metric spaces are introduced and a fixed point theorem is proved. There was no discussion on the completion of partial b-metric spaces as well as topological properties of these spaces. So the completion problem for partial b-metric spaces using the quotient space of Cauchy sequences remains an open problem.

It is the purpose of this paper to introduce the notion of a strong partial b-metric space, construct a 0-Cauchy completion of a strong partial b-metric space and show that the 0-completion is unique up to isometry. We will also discuss the relationship between strong partial b-metric spaces and strong b-metric spaces as well as their completions. In one of the results we will show that the 0-Cauchy completion of a strong partial b-metric space is weakly isometric to the completion of a strong b-metric space.

2. Preliminaries

We introduce:

Definition 2.1. Let X be a nonempty set. A map $\sigma : X \times X \to [0, \infty)$ is a **strong partial b-metric** on X if for all $x, y, z \in X$, and $\alpha \geq 1$ the following conditions hold:

(i) $x = y$ if $\sigma(x, x) = \sigma(x, y) = \sigma(y, y)$;

(ii) $\sigma(x, x) \leq \sigma(x, y)$;

(iii) $\sigma(x, y) = \sigma(y, x)$;

(iv) $\sigma(x, z) \leq \sigma(x, y) + \alpha\sigma(y, z) - \sigma(y, y)$.

The triple (X, α, σ) is called a **strong partial b-metric space**.

Note that if $\sigma(x, y) = 0$, then from (i) and (ii) we obtain that $x = y$. But if $x = y$, then $\sigma(x, y)$ is not necessarily 0.

Remark 2.2. In Definition 2.1, when $\alpha = 1$, then (X, α, σ) is a partial metric space. Every partial metric space is a strong partial b-metric space but not conversely.

Definition 2.3. [4] Let X be a nonempty set. A map $\sigma : X \times X \to [0, \infty)$ is a **strong b-metric** on X if for all $x, y, z \in X$, and $\alpha \geq 1$ the following conditions hold:

(i) $x = y$ if and only if $\sigma(x, y) = 0$;

(ii) $\sigma(x, y) = \sigma(y, x)$;

(iii) $\sigma(x, z) \leq \sigma(x, y) + \alpha\sigma(y, z)$.

The triple (X, α, σ) is called a **strong b-metric space**.

Remark 2.4. (i) Every metric space is a strong b-metric space but the converse is not necessarily true.

(ii) Every strong b-metric space is a strong partial b-metric space but not conversely.

That is: metric space $\Rightarrow$ strong b-metric space $\Rightarrow$ strong partial b-metric space.

Definition 2.5. Let (X, α, σ) be a strong partial b-metric space. Then

(i) a sequence $\{x_n\}$ in (X, α, σ) **converges** to a point $x \in X$ if $\sigma(x, x) = \lim_n \sigma(x_n, x) = \lim_n \sigma(x_n, x_n)$.

(ii) a sequence $\{x_n\}$ in (X, α, σ) is **Cauchy** if the $\lim_{n,m} \sigma(x_n, x_m)$ exists and finite.

(iii) a strong partial b-metric space (X, α, σ) is **Cauchy complete** if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$.

Definition 2.6. Let (X, α, σ) be a strong partial b-metric space. Then

(i) a sequence $\{x_n\}$ in (X, α, σ) is 0-**Cauchy** if $\lim_{n,m} \sigma(x_n, x_m) = 0$.

(ii) a strong partial b-metric space (X, α, σ) is 0-**Cauchy complete** if every 0-Cauchy sequence in X converges to a point $x \in X$ and $\sigma(x, x) = 0$.

Remark 2.7. Note the following:

(i) every 0-Cauchy sequence is a Cauchy sequence but the converse does not necessarily hold.

(ii) every Cauchy complete strong partial b-metric space is 0-Cauchy complete but the converse is not necessarily true.

(iii) a subsequence of a 0-Cauchy sequence is also a 0-Cauchy sequence.

Example 2.1. Let $X = \{1, 2, 3\}$. Define a map $\sigma : X \times X \to [0, \infty)$ by $\sigma(x, y) = max\{x, y\} + d(x, y)$, where $d(x, y) = 0$ for all $x = y$ and $d(1, 2) = d(2, 1) = 2, d(1, 3) = d(3, 1) = 6, d(2, 3) = d(3, 2) = 1$. Then (X, σ) is neither a partial metric space nor a strong b-metric space. However, we see that (X, α, σ) is a strong partial b-metric space with $\alpha = 4$.

The following proposition shows us on how to construct further examples of strong partial b-metric spaces.

Proposition 2.8. *Let X be a set and a map $\sigma : X \times X \to [0, \infty)$ be defined $\sigma(x, y) = p(x, y) + d(x, y)$, for all $x, y \in X$, where p is a partial metric on X and d a strong b-metric on X, with a parameter α. Then (X, α, σ) is a strong partial b-metric space with parameter α.*

Proof. We note that $\sigma(x, x) = p(x, x)$, and thus $\sigma(x, x) \le \sigma(x, y)$ for all $x, y \in X$. Also, $\sigma(x, y) = \sigma(x, x) = \sigma(y, y)$ if and only if $x = y$. Clearly $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in X$. Finally: $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$ while $d(x, y) \le d(x, z) + \alpha d(z, y)$. So,

$$\sigma(x, y) \le p(x, z) + p(z, y) - p(z, z) + d(x, z) + \alpha d(z, y).$$

Then

$$\sigma(x, y) \le [p(x, z) + d(x, z) + \alpha p(z, y) + \alpha d(z, y) - p(z, z) - d(z, z)].$$

Therefore, $\sigma(x, y) \le \sigma(x, z) + \alpha\sigma(z, y) - \sigma(z, z)$. Therefore (X, α, σ) is a strong partial b-metric space. □

3. Completions

Definition 3.1. Let (X, α, σ) be a strong partial b-metric space and A be a subset of X. We say that A is **sequentially dense** in X if for $x \in X$, there is a sequence $\{a_n\}$ in A that converges to x.

Definition 3.2. Let (X, α, σ_X) and (Y, β, σ_Y) be strong partial b-metric spaces. A map $T : (X, \alpha, \sigma_X) \to (Y, \beta, \sigma_Y)$ is an isometry if $\sigma_Y(Tx, Ty) = \sigma_X(x, y)$ for all $x, y \in X$.

It is worth mentioning that a strong partial b-metric space can be regarded as a topological space. Let $x \in X$, and $\epsilon > 0$. Set $B(x, \sigma, \epsilon) = \{y \in X : \sigma(x, y) < \epsilon + \sigma(x, x)\}$. We say that $A \subseteq X$, is open in X if for each $a \in A$, there exists $\epsilon > 0$ such that $B(a, \sigma, \epsilon) \subset A$. Then for $x \in X$ the set $\{B(x, \sigma, \epsilon)\}$ forms a **subbasis** of a topology on X. The topology on X induced by σ is denoted by $\tau(\sigma)$. Topological properties of strong partial b-metric spaces will be discussed elsewhere. It is important to note that, for $x \in X$, and $\epsilon > 0$, the set $B(x, \sigma, \epsilon)$ in a strong partial b-metric space (X, α, σ) is open and the topology $\tau(\sigma)$ induced by σ on X is an asymmetric topology.

Definition 3.3. Let (X, α, σ) be a strong partial b-metric space and $\{x_n\}$ be a sequence in X. We will say that a sequence $\{x_n\}$ converges to $x \in X$, with respect to $\tau(\sigma)$ if the following holds $lim_n \sigma(x_n, x) = \sigma(x, x)$.

Obviously, a sequence $\{x_n\}$ in a strong partial b-metric space converges to $x \in X$, implies that $\{x_n\}$ converges to x with respect to $\tau(\sigma)$, but not conversely. This follows from the fact that a sequence $\{x_n\}$

converges to x if for $\epsilon > 0$ there exists a natural number N such that $\sigma(x_n, x) < \epsilon + \sigma(x, x)$ and $\sigma(x_n, x) < \epsilon + \sigma(x_n, x_n)$ for all $n \geq N$.

Definition 3.4. Let (X, α, σ) be a strong partial b-metric space and A be a subset of X. We say that A is **symmetrically dense** in X if for each $x \in X$, and $\epsilon > 0$ there exists $a \in A$ such that $x \in B(a, \sigma, \epsilon)$ and $a \in B(x, \sigma, \epsilon)$.

Proposition 3.5. *Let (X, α, σ) be a strong partial b-metric space and A be a subset of X. Then A is sequentially dense if and only if A is symmetrically dense.*

Proof. Suppose that $A \subseteq X$ is symmetrically dense. Let $x \in X$. For $\epsilon_1 = 1$, find $a_1 \in A$, such that $x \in B(a_1, \sigma, \epsilon_1)$ and $a_1 \in B(x, \sigma, \epsilon_1)$. For $\epsilon_2 = \frac{1}{2}$, find $a_2 \in A$, such that $a_2 \in B(x, \sigma, \epsilon_2)$ and $x \in B(a_2, \sigma, \epsilon_2)$, we proceed this way, for ϵ_n, find $a_n \in A$, such that $a_n \in B(x, \sigma, \epsilon_n)$ and $x \in B(a_n, \sigma, \epsilon_n)$. Clearly $\{a_n\}$ is a sequence in A, we now show that $\{a_n\}$ converges to x.

So $\sigma(a_n, x) < \epsilon_n + \sigma(x, x)$ and $\sigma(a_n, x) < \epsilon_n + \sigma(a_n, a_n)$. Now $lim_n \epsilon_n = 0$, thus

$$\sigma(x, x) \leq lim_n \sigma(a_n, x) \leq \sigma(x, x).$$

So $\sigma(x, x) = lim_n \sigma(a_n, x)$. Also,

$$\sigma(a_n, x) \leq \epsilon_n + \sigma(a_n, a_n) \leq \epsilon_n + \sigma(a_n, x),$$

it follows that $lim_n \sigma(a_n, x) \leq lim_n \epsilon_n + lim_n \sigma(a_n, a_n) \leq lim_n \epsilon_n + lim_n \sigma(a_n, x)$. We get

$$\sigma(x, x) \leq lim_n \sigma(a_n, a_n) \leq \sigma(x, x).$$

Hence $lim_n \sigma(a_n, a_n) = \sigma(x, x)$. This shows that $\{a_n\}$ converges to x.

Conversely, suppose that $x \in X$, and we find sequence $\{a_n\}$ in A such that $\{a_n\}$ converges to x. Since A is symmetrically dense in X. the following holds: For $\epsilon > 0$ and a natural number N, $x \in B(a_n, \sigma, \epsilon)$ and $a_n \in B(x, \sigma, \epsilon)$, for all $n \geq N$. Let $a = n_k$, with $k \geq N$. Then $a \in A$, and $a \in B(x, \sigma, \epsilon)$ and $x \in B(a, \sigma, \epsilon)$. □

Definition 3.6. Let (X, α, σ) be a strong partial b-metric space. A subset $A \subseteq X$, is said to be dense in X if for every $x \in X$ there exists a sequence $\{a_n\}$ in A such that a_n converges to x with respect to $\tau(\sigma)$.

Let (X, α, σ) be a strong partial b-metric space. If a subset A of X is sequentially (symmetrically) dense, then it is dense in X and not conversely.

For strong b metric spaces we have:

Theorem 3.7. *Let (X, α, σ) be a strong b-metric space and A be a subset of X. Then the following statements are equivalent:*

(i) A is dense in X;

(ii) A is symmetrical dense in X;

(iii) A is sequentially dense in X.

Definition 3.8. Let (X, α, σ) be a strong partial b-metric space. We say that a strong partial b-metric space $(\bar{X}, \alpha, \bar{\sigma})$ is a 0-**Cauchy completion** of (X, α, σ) if

(*i*) $(\bar{X}, \alpha, \bar{\sigma})$ is 0-Cauchy complete;

(*ii*) $X \subseteq \bar{X}$, and $\bar{\sigma}|_{X \times X} = \sigma$;

(*iii*) there exists $T : (X, \alpha, \sigma) \to (\bar{X}, \alpha, \bar{\sigma})$, such that T is an isometry.

(*iv*) TX is a sequentially dense in $\bar{X}$.

Theorem 3.9. *Every strong partial b-metric space (X, α, σ) admits a 0-Cauchy completion $(\bar{X}, \alpha, \bar{\sigma})$.*

Proof. (i) Let $\mathcal{C} = \{x_n : \{x_n\}$ be a 0-Cauchy sequence in $(X, \alpha, \sigma)\}$. For $x_n, y_n \in \mathcal{C}$, write

$$x_n \sim y_n \Leftrightarrow \lim_n \sigma(x_n, x_n) = \lim_n \sigma(y_n, y_n) = 0.$$

Then $\sim$ is an equivalence relation on $\mathcal{C}$.
(ii) Next, let $\mathcal{K} = \{x$: where $\{x\}$ is an eventually constant sequence which is not a 0-Cauchy sequence in $(X, \alpha, \sigma)\}$. For $x, y \in \mathcal{K}$, write

$$x \sim y \Leftrightarrow x = y.$$

Then $\sim$ is an equivalence relation on $\mathcal{K}$.

(iii) If $x \in \mathcal{K}$ and $x_n \in \mathcal{C}$, write

$$x \sim x_n \Leftrightarrow \lim_n \sigma(x, x_n) = 0.$$

Put $\bar{X}$ to be the set of all the equivalence classes in $\mathcal{K}$ together with the set of all the equivalence classes in $\mathcal{C}$, that is,

$$\bar{X} = \{[\{x\}] : x \in \mathcal{K}\} \cup \{[\{x_n\}] : \{x_n\} \in \mathcal{C}\}.$$

Now for every $\bar{x}, \bar{y} \in \bar{X}$, define $\bar{\sigma} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{\sigma}(\bar{x}, \bar{y}) = \lim_n \sigma(x_n, y_n),$$

where $\bar{x} = [\{x_n\}]$ and $\bar{y} = [\{y_n\}]$. Then: (a) $\bar{\sigma}$ is well defined on $\bar{X}$. Let $\bar{x}, \bar{y} \in \bar{X}$ then $\bar{\sigma}(\bar{x}, \bar{y}) = lim_n \sigma(x_n, y_n)$, where $\bar{x} = [\{x_n\}]$ and $\bar{y} = [\{y_n\}$, we will show that $lim_n \sigma(x_n, y_n)$ exists. Case (1): If $x_n, y_n \in \mathcal{C}$. Then

$$\sigma(x_n, y_n) \leq \alpha\sigma(x_n, x_m) + \sigma(x_m, y_m) + \alpha\sigma(y_m, y_n) - \sigma(x_m, x_m) - \sigma(y_m, y_m),$$

so,

$$\sigma(x_n, y_n) - \sigma(x_m, y_m) \leq \alpha[\sigma(x_n, x_m) + \sigma(y_m, y_n)] - \sigma(x_m, x_m) - \sigma(y_n, y_n).$$

Similarly, we have

$$\sigma(x_m, y_m) - \sigma(x_n, y_n) \leq \alpha[\sigma(x_n, x_m) + \sigma(y_n, y_m)] - \sigma(x_m, x_m) - \sigma(y_n, y_n).$$

and therefore

$$|\sigma(x_n, y_n) - \sigma(x_m, y_m)| \leq \alpha[\sigma(x_n, x_m) + \sigma(y_n, y_m)] - \sigma(x_m, x_m) - \sigma(y_n, y_n).$$

Now $\sigma(x_n, x_n) \leq \sigma(x_m, x_n)$ for all natural m and n. Now

$$lim_{m,n} \alpha[\sigma(x_n, x_m) + \sigma(y_n, y_m)] - \sigma(x_m, x_m) - \sigma(y_n, y_n) = 0,$$

and so $lim_{m,n}|\sigma(x_n, y_n) - \sigma(x_m, y_m)| = 0$. We conclude that $lim_n \sigma(x_n, y_n)$, exists in $[0, \infty)$. Case (2), If and $x_n, y_n \in \mathcal{K}$ then $\bar{\sigma}(\bar{x}, \bar{y}) = \sigma(x, y)$ which exists. Case (3). Let $x_n \in \mathcal{C}$ and $y_n \in \mathcal{K}$. Then $\bar{\sigma}(\bar{x}, \bar{y}) = lim_n \sigma(x_n, y)$. So,

$$\sigma(x_n, b) - \sigma(x_m, b) \leq \alpha\sigma(x_n, x_m) - \sigma(x_m, x_m)$$

and $lim_{m,n}(\sigma(x_n, b) - \sigma(x_m, b)) = 0$. Let $a_n = \sigma(x_n, b)$ for each $n \in \mathbb{N}$. We get $lim_{m,n}(a_n - a_m) = 0$. We also have,

$$a_m - a_n \leq \alpha\sigma(x_n, x_m) - \sigma(x_n, x_n) \to 0,$$

as $m, n \to \infty$. Therefore, $lim_{m,n}|a_m - a_n| = 0$, and thus $lim_n\sigma(x_n, b)$ exists. Case (4), Let $x_n \in \mathcal{K}$ and $y_n \in \mathcal{C}$. The proof is similar to that of case (3). So $\bar{\sigma}$ is well-defined. We will show that (b) $(\bar{X}, \alpha, \bar{\sigma})$ is a strong partial b-metric space. Let $\bar{x}, \bar{y}, \bar{z} \in \bar{X}$ and $\bar{x} = [\{x_n\}], \bar{y} = [\{y_n\}]$ and $\bar{z} = [\{z_n\}]$. We only consider the case where $x_n, y_n, z_n \in \mathcal{C}$, the other possible cases are similar and easy to show. Certainly, the following holds: $\bar{\sigma}(\bar{x}, \bar{y}) \geq 0$, as $\sigma(x, y) \geq 0$ for all $x, y \in X$. Also, $\sigma(x, x) \leq \sigma(x, y)$ for all $x, y \in X$, so $\bar{\sigma}(\bar{x}, \bar{x}) \leq \bar{\sigma}(\bar{x}, \bar{y})$, for all $\bar{x}, \bar{y} \in \bar{X}$. Furthermore, $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in X$, and thus $\bar{\sigma}(\bar{x}, \bar{y}) = \bar{\sigma}(\bar{y}, \bar{x})$. Finally,

$$\bar{\sigma}(\bar{x}, \bar{y}) = lim_n\sigma(x_n, y_n)$$

and

$$lim_n\sigma(x_n, y_n) \leq lim_n(\sigma(x_n, z_n) + \alpha\sigma(z_n, y_n) - \sigma(z_n, z_n)).$$

Thus $\bar{\sigma}(\bar{x}, \bar{y}) = lim_n\sigma(x_n, z_n) + \alpha lim_n\sigma(z_n, y_n) - lim_n\sigma(z_n, z_n)$. It follows that

$$\bar{\sigma}(\bar{x}, \bar{y}) \leq \bar{\sigma}(\bar{x}, \bar{z}) + \alpha\bar{\sigma}(\bar{z}, \bar{y}) - \bar{\sigma}(\bar{z}, \bar{z}).$$

(c) $T : (X, \alpha, \sigma) \to (\bar{X}, \alpha, \bar{\sigma})$ defined by $T(x) = [\{x\}]$ is an isometry. Note that

$$\bar{\sigma}(Tx, Ty) = lim_n\sigma(x, y) = \sigma(x, y),$$

for all $x, y \in X$. (d) Next we show that TX is sequentially dense in $\bar{X}$. Let $\bar{x} \in \bar{X}$. If $\bar{x} \in \mathcal{K}$ then we are done, so assume that $\bar{x} = [\{x_n\}]$ and $\{x_n\}$ is a 0-Cauchy sequence in (X, α, σ). That is, $x_n \in \mathcal{C}$, so $lim_{m,n}\sigma(x_m, x_n) = 0$. For $\epsilon > 0$ there exists a natural number N, such that $\sigma(x_m, x_n) < \frac{\epsilon}{3}$, for all $m, n \geq N$. Thus for $m, n \geq N, \sigma(x_n, x_m) < \frac{\epsilon}{3}$, and $\sigma(x_{n_0}, x_0) < \frac{\epsilon}{3}$. Let $x = x_{n_0}$. Then $x \in TX$, since $Tx = [\{x_{n_0}\}] \subseteq \bar{X}$, and $lim_n\bar{\sigma}([\{x_{n_0}\}], \bar{x}) = \bar{\sigma}(\bar{x}, \bar{x}) = lim_n\bar{\sigma}([\{x_{n_0}\}], [\{x_{n_0}\}]) = 0$. This shows that TX is sequentially dense in $\bar{X}$. (e) Finally, we show that $(\bar{X}, \alpha, \bar{\sigma})$ is 0-Cauchy complete. Let $\{\bar{x}_n\}$ be a 0-Cauchy sequence in $(\bar{X}, \alpha, \bar{\sigma})$. Then $x_n \in \mathcal{C}$ for all $n \in \mathbb{N}$ and $\bar{x}_n = [\{x_i^n\}_i]$ for $\{x_n\}_i \in \mathcal{C}$ and $lim_{m,n}\bar{\sigma}(\bar{x}_m, \bar{x}_n) = 0$. Since TX is sequentially dense in $\bar{X}$, for each $n \in \mathbb{N}$, there exists $y_n \in X$ such that $\{y_n\}$ converges to $[\{\bar{x}_n\}]$, that is, $\bar{\sigma}(\bar{x}_n, y_n) < \frac{1}{n\alpha} + \bar{\sigma}(\bar{x}_n, \bar{x}_n)$. Now

$$\sigma(y_n, y_m) = \bar{\sigma}(Ty_n, Ty_m)$$

and $\bar{\sigma}(Ty_n, Ty_m) \leq \alpha\bar{\sigma}(Ty_n, \bar{x}_n) + \bar{\sigma}(\bar{x}_n, \bar{x}_m) + \alpha\bar{\sigma}(\bar{x}_m, Ty_m) - \bar{\sigma}(\bar{x}_n, \bar{x}_n) - \bar{\sigma}(\bar{x}_m, \bar{x}_m)$. So, $\bar{\sigma}(Ty_n, Ty_m) \leq \frac{1}{n} + lim_m\bar{\sigma}(\bar{x}_n, \bar{x}_m) + \frac{1}{m} - \sigma(x_n, x_n) - \sigma(x_m, x_m)$. Since $\bar{\sigma}(\bar{x}, \bar{x}) \leq \bar{\sigma}(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in \bar{X}$, and

$$lim_{m,n}[\bar{\sigma}(\bar{x}_n, \bar{x}_n) - \bar{\sigma}(\bar{x}_m, \bar{x}_m)] \leq lim_{m,n}[\bar{\sigma}(\bar{x}_m, \bar{x}_n) + \bar{\sigma}(\bar{x}_m, \bar{x}_m)] = 0.$$

We get $lim_{m,n}\sigma(y_m, y_n) = 0$. So $\{y_n\}$ is a 0-Cauchy sequence in (X, α, σ). Let $\bar{y} = [\{y_n\}]$. Then $\bar{y} \in \bar{X}$. Next,

$$\bar{\sigma}(\bar{x}_n, \bar{y}) \leq \alpha\bar{\sigma}(\bar{x}_n, Ty_n) + \bar{\sigma}(Ty_n, \bar{y}) - \bar{\sigma}(Ty_n, Ty_n)$$

and

$$\alpha\bar{\sigma}(\bar{x}_n, Ty_n) + \bar{\sigma}(Ty_n, \bar{y}) - \bar{\sigma}(Ty_n, Ty_n) < \frac{1}{n} + lim_m \bar{\sigma}(y_n, y_m) - \bar{\sigma}(y_n, y_n).$$

It follows that $lim_n \bar{\sigma}(\bar{x}_n, \bar{y}) = 0$, also, we have $\bar{\sigma}(\bar{y}, \bar{y}) = 0$ and $\bar{\sigma}(\bar{x}_n, \bar{x}_n) = 0$. So $\bar{x}_n$ converges to $\bar{y}$ in $(\bar{X}, \alpha, \sigma)$. So $(\bar{X}, \alpha, \sigma)$ is complete. □

Theorem 3.10. *Let (X, α, σ) be a strong partial b-metric space. The strong partial b-metric 0-Cauchy completion of (X, α, σ) is unique up to isometry.*

Proof. Let $(\bar{X}, \alpha, \bar{\sigma})$ and $(\acute{X}, \alpha, \acute{\sigma})$ be the two 0-Cauchy completions of (X, α, σ). Then there exist isometric embeddings $T_1 : X \to \bar{X}$ and $T_2 : X \to \acute{X}$. For each $\bar{x} \in \bar{X}$, we can find $\{x_n\}$ in X such that $T_1 x_n$ converges to $\bar{x}$. Also $T_2 x_n$ converges to some $\acute{x} \in \acute{X}$. Define $\varphi : \bar{X} \to \acute{X}$, by $\varphi(\bar{x}) = \acute{x}$. The map φ is bijective and an isometry. □

If (X, α, σ) is a strong b-metric space then its completion [Theorem 2.2 in [1]] and the 0-Cauchy completion [Theorem 3.9] of (X, α, σ) coincide.

4. Strong partial *b*-metric spaces, strong *b*-metric spaces and completions

We start with the following result:

Proposition 4.1. *Let (X, α, σ) be a strong partial b-metric space. Define $d_\sigma : X \times X \to [0, \infty)$ by*

$$d_\sigma(x, y) = \begin{cases} \sigma(x, y) & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y. \end{cases}$$

Then, (X, α, d_σ) is a strong b-metric space.

Proof. We first note that $d_\sigma(x, y) = 0$ if and only if $x = y$, also, $d_\sigma(x, y) = d_\sigma(y, x)$ for all $x, y \in X$. Now let $x, y, z \in X$. We consider the following cases: (i) If $x = y = z$, then $d_\sigma(x, y) = \alpha d_\sigma(x, z) + d_\sigma(z, y)$. Case (ii): Now suppose that $x = y$, but $y \neq z$. Then $x \neq z$. So we have $0 = d_\sigma(x, y) \leq \sigma(x, y) \leq \alpha\sigma(x, z) + \sigma(z, y) - \sigma(z, z) \leq \alpha\sigma(x, z) + \sigma(z, y) = \alpha d_\sigma(x, z) + d_\sigma(z, y)$. Case (iii) Assume that $x \neq y$, and $z = x$. Then $z \neq y$. So, $d_\sigma(x, y) = \sigma(x, y) \leq \alpha\sigma(x, z) + \sigma(z, y) - \sigma(z, z) \leq \sigma(z, y) = \alpha d_\sigma(x, z) + d_\sigma(z, y)$. Case (iv): Suppose that $x \neq y$, and $y = z$. Then $x \neq z$. We get $d_\sigma(x, y) = \sigma(x, y) \leq \alpha\sigma(x, z) + \sigma(z, y) - \sigma(z, z) = \alpha\sigma(x, z) + \sigma(z, y) = \alpha d_\sigma(x, z) \leq \alpha d_\sigma(x, z) + d_\sigma(z, y)$. Case (v), Suppose that $x \neq y$ and $y \neq z$. Then $d_\sigma(x, y) = \sigma(x, y) \leq \alpha\sigma(x, z) + \sigma(z, y) - \sigma(z, z) \leq \alpha\sigma(x, z) + \sigma(z, y) = \alpha d_\sigma(x, z) + d_\sigma(z, y)$. □

In the next result, given a strong partial b-metric space (X, α, σ) we shall refer to the strong b-metric space (X, α, d_σ) from Proposition 4.1 as the associated strong b-metric space.

Lemma 4.2. *Let (X, α, σ) be a strong partial b-metric space and (X, α, d_σ) be the associated strong b-metric space. If a sequence $\{x_n\}$ is a 0-Cauchy sequence in (X, α, σ) then it is a Cauchy sequence in (X, α, d_σ).*

Theorem 4.3. *Let the strong partial b-metric space (X, α, σ) be 0-Cauchy complete. Then the associated strong b-metric space (X, α, d_σ) is Cauchy complete.*

Definition 4.4. A map $T : (X, \alpha, \sigma_X) \to (Y, \beta, \sigma_Y)$ between strong partial b-metric spaces is a **weak isometry** if $\sigma_Y(Tx, Ty) = \sigma_X(x, y)$ for all $x \neq y \in X$.

Definition 4.5. Let (X, α, σ) be a strong partial b-metric space. We say that a strong b-metric space $(\bar{X}, \beta, \bar{\sigma})$ is a **completion** of (X, α, σ) if

(i) $(\bar{X}, \beta, \bar{\sigma})$ is a complete;

(ii) $X \subseteq \bar{X}$, and $\bar{\sigma}|_{X\times X} = \sigma$, for $x \neq y$;

(iii) there exists $T : (X, \alpha, \sigma) \to (\bar{X}, \beta, \bar{\sigma})$, such that T is a weak isometry.

Now given a strong partial b-metric space (X, α, σ) we let (X, α, d_σ) be a strong b-metric space as in Proposition 4.1 and $(\bar{X}, \alpha, \bar{d_\sigma})$ be the strong b-metric completion of (X, α, d_σ) [1, Theorem 2.2].

Theorem 4.6. *Every strong partial b-metric space (X, α, σ) admits a strong b-metric completion.*

Proof. Let $(\bar{X}, \alpha, \bar{d_\sigma})$ be the completion of (X, α, d_σ), [1, Theorem 2.2]. Then (X, α, d_σ) is sequentially dense in $(\bar{X}, \alpha, \bar{d_\sigma})$. In particular, there is an isometry $T : (X\alpha, d_\sigma) \to (\bar{X}, \alpha, \bar{d_\sigma})$ defined by $Tx = [\{x\}]$, where, $[\{x\}] \in \bar{X}$ for $x \in X$. Then for the maps $I : (X, \alpha, \sigma) \to (X, \alpha, d_\sigma)$ with $I(x) = x$, for $x \in X$ and $T : (X, \alpha, d_\sigma) \to (\bar{X}, \alpha, \bar{d_\sigma})$, clearly, $T \circ I : X \to \bar{X}$ is a weak isometry. This shows that $(\bar{X}, \alpha, \bar{d_\sigma})$ is a strong b-metric completion of (X, α, σ). $\square$

Since the completion of a strong b-metric space is unique up isometry [1].

Theorem 4.7. *The strong b-metric completion of a strong partial b-metric space is unique up to a weak isometry.*

Definition 4.8. Two strong partial b-metric spaces (X, α, σ_X) and (Y, β, σ_Y) are said to be weakly isometric if there exists a weak isometry $T : (X, \alpha, \sigma_X) \to (Y, \beta, \sigma_Y)$ between them.

Theorem 4.9. *Let (X, α, σ) be a strong partial b-metric space. Then the 0-Cauchy completion $(\bar{X}, \alpha, \bar{\sigma})$ and strong b-metric completion $(\bar{X}, \alpha, \bar{\sigma})$ of (X, α, σ) are weakly isometric.*

Proof. Let $(\bar{X}, \alpha, \bar{d_\sigma})$ and $(\bar{X}, \alpha, \bar{\sigma})$ be the 0-Cauchy completion and strong b-metric completion of (X, α, σ), respectively, and the maps $I : (X, \alpha, \sigma) \to (X, \alpha, d_\sigma), S : (X, \alpha, \sigma) \to (\bar{X}, \alpha, \bar{\sigma})$, and $H : (X, \alpha, d_\sigma) \to (\bar{X}, \alpha, \bar{d_\sigma})$, are an identity, a weak isometric embeding and a weak isometric embedding, respectively. The map $T : (\bar{X}, \alpha, \bar{\sigma}) \to (\bar{X}, \alpha, \bar{d_\sigma})$ defined by $T = H \circ I \circ S^{-1}$ is a weak isometry. $\square$

In concluding the paper, we wish to highlight two important points.

(a) Given a strong partial b-metric space (X, α, σ) one can also obtain its 0-Cauchy completion $(\bar{X}, \alpha, \bar{\sigma})$ by using the strong b-metric completion. Let $(\bar{X}, \alpha, \bar{d_\sigma})$ be the strong b-completion of (X, α, σ) from Theorem 4.6. We construct a strong partial b-metric $\bar{\sigma}$ on $\bar{X}$ directly from $\bar{d_\sigma}$, in the following way:

$$\bar{\sigma}(\bar{x}, \bar{y}) = \begin{cases} \bar{d_\sigma}(\bar{x}, \bar{y}) & \text{if} \quad \bar{x} \neq \bar{y}; \\ \sigma(x, x) & \text{if} \quad \bar{x} = x = \bar{y} \in X; \\ 0 & \text{if} \quad \bar{x} = x = \bar{y} \notin X. \end{cases}$$

Then, $(\bar{X}, \alpha, \bar{\sigma})$ is a 0-complete strong partial b-metric space, this follows from the fact that $(\bar{X}, \alpha, \bar{d_\sigma})$ is a complete strong b-metric space. For each $x \in X$, a map $T : X \to \bar{X}$ defined by $Tx = [\{x\}]$ is an isometry. This further confirms Theorem 3.9.

(b) Given a contraction map T on a complete strong partial b-metric space (X, α, σ), that is, $T : X \to X$ and there exists $k \in (0, \frac{1}{\alpha})$ such that $\sigma(Tx, Ty) \leq k\sigma(x, y)$. Then we can show that T has a unique fixed point. This follows from the fact that (X, α, σ) is also a complete partial b-metric space [7], and

Proposition 4.1, enables us to consider T as a contraction on a b-metric space (X, α, d_σ), which is known (see [3] and [2, Corollary 2.1]) to have a unique fixed point.

Hence,

Theorem 4.10. *Let (X, α, σ) be a complete strong partial b-metric space and $T : X \to X$ be a self-map. Suppose that there exists $k \in (0, \frac{1}{\alpha})$ such that $\sigma(Tx, Ty) \leq k\sigma(x, y)$, for all $x, y \in X$. Then T has a unique fixed point.*

5. Conclusion

The paper introduced strong partial b-metric spaces and provided some fundamental properties of these spaces. The strong partial b-metric on X has better topological properties than a partial b-metric, hence, one advantage discussed in the paper is the completion problem for strong partial b-metric spaces. We also related the completion problem of the strong partial b-spaces to that of strong b-metric spaces. The authors are also interested in studying more topological properties of these spaces in the future. The completion problem for partial b-metric spaces is an open problem.

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