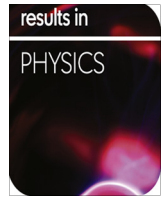


# TUTDoR

## Conservation laws for perturbed solitons in optical metamaterials.

Item Type	Article
Authors	Biswas, Anjan;Yasar, Emrullah;Yıldırım, Yakup;Triki, Houria;Zhou, Qin;Moshokoa, Seithuti P.;Belic, Milivoj
Publisher	Elsevier
Rights	Attribution-NonCommercial-ShareAlike 4.0 International
Download date	2026-05-09 17:34:53
Item License	<a href="http://creativecommons.org/licenses/by-nc-sa/4.0/">http://creativecommons.org/licenses/by-nc-sa/4.0/</a>
Link to Item	<a href="https://hdl.handle.net/20.500.14519/1065">https://hdl.handle.net/20.500.14519/1065</a>



## Conservation laws for perturbed solitons in optical metamaterials

Anjan Biswas<sup>a,b,c</sup>, Emrullah Yaşar<sup>d,\*</sup>, Yakup Yıldırım<sup>d</sup>, Houria Triki<sup>e</sup>, Qin Zhou<sup>f</sup>,  
Seithuti P. Moshokoa<sup>c</sup>, Milivoj Belic<sup>g</sup>

<sup>a</sup> Department of Physics, Chemistry and Mathematics, Alabama A&M University, Normal, AL 35762, USA

<sup>b</sup> Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

<sup>c</sup> Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria 0008, South Africa

<sup>d</sup> Department of Mathematics, Faculty of Arts and Sciences, Uludag University, 16059 Bursa, Turkey

<sup>e</sup> Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria

<sup>f</sup> School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan 430212, PR China

<sup>g</sup> Science Program, Texas A & M University at Qatar, PO Box 23874, Doha, Qatar



### ARTICLE INFO

#### Article history:

Received 30 November 2017

Accepted 25 December 2017

Available online 11 January 2018

#### Keywords:

Conservation laws

Lie symmetry

Optical metamaterials

### ABSTRACT

The conservation laws for the dynamics of soliton propagation through optical metamaterials are derived by the aid of Lie symmetry analysis. The proposed model will be studied with two forms of nonlinearity. They are Kerr law and parabolic law.

© 2018 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

### Introduction

Performance enhancement is one of the key issues in the study of soliton propagation through optical waveguides. In this context, there are various kinds of waveguides that has been lately considered [1–10]. A few of them are optical fibers, couplers, PCF, optical metamaterials and metasurfaces. This paper focuses on a particular issue in the soliton propagation dynamics through optical metamaterials. It is the aspect of conservation laws. No study of a physical phenomena is complete without a listing of conserved quantities. While there are several papers visible in this area, it was never reported earlier about conserved quantities for soliton propagation through such metamaterials. This paper thus fills in the gap. The conservation laws for optical metamaterials are presented when a few Hamiltonian type perturbation terms are taken into account. There are two types of nonlinearity that successfully retrieves these conserved densities by the aid of Lie symmetry analysis. These are Kerr law and parabolic law. The conserved quantities are subsequently computed from bright 1-soliton solutions that were reported earlier. These are detailed in the following sections.

### Governing equation

The dynamics of solitons in optical metamaterials is governed by the nonlinear Schrödinger's equation (NLSE) which in the dimensionless form is given by [2,10]

$$iq_t + aq_{xx} + F(|q|^2)q = i\alpha q_x + i\lambda(|q|^2 q)_x + iv(|q|^2)_x q + \theta_1(|q|^2 q)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q_{xx}^* \quad (1)$$

Eq. (1) is the NLSE that is studied in the context of metamaterials. Here in (1),  $a$  and  $b$  are the group velocity dispersion and the self-phase modulation terms respectively. This pair produces the delicate balance between dispersion and nonlinearity that accounts for the formation of the stable solitons. On the right-hand side  $\lambda$  represents the self-steepening term in order to avoid the formation of shocks and  $v$  is the nonlinear dispersion, while  $\alpha$  represents the inter-modal dispersion. This arises from the fact that group velocity of light in multi-mode fibers depends on chromatic dispersion as well as the propagation mode involved. Next,  $\theta_j$  for  $j = 1, 2, 3$  are the perturbation terms that appear in the context of metamaterials. Finally, the independent variables are  $x$  and  $t$  that represent spatial and temporal variables respectively with the dependent variable  $q(x, t)$  being the complex-valued wave profile.

\* Corresponding author.

E-mail address: [emrullah.yasar@gmail.com](mailto:emrullah.yasar@gmail.com) (E. Yaşar).

The real-valued algebraic functional  $F$  must possess smoothness of the complex-valued function  $F(|q|^2)q : C \rightarrow C$ . Treating the complex plane  $C$  as two-dimensional linear space  $R^2$ , the function  $F(|q|^2)q$  is  $k$  times continuously differentiable provided

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k((-n, n) \times (-m, m); R^2). \tag{2}$$

**Conserved densities**

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence [5]. Firstly, if  $(T^t, T^x)$  is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{3}$$

along the solutions of the differential equation (de = 0). For a scalar pde  $E(x, t, q, q_{(1)}, \dots) = 0$ , if there exists a nontrivial differential function  $Q$ , called a ‘multiplier’, such that

$$E_q[QE] = 0$$

then  $QE$  is a total divergence, i.e.,

$$QE = D_t T^t + D_x T^x$$

for some (conserved) vector  $(T^t, T^x) - E_q$  is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier  $Q$  leads to a conserved vector determined by, amongst other things, a Homotopy operator.

$$E^1(x, t, u, v, u_{(1)}, v_{(1)}, \dots) = 0, \quad E^2(x, t, u, v, u_{(1)}, v_{(1)}, \dots) = 0 \tag{4}$$

For a system (4),  $Q = (Q^1, Q^2)$ , say, so that

$$Q^1(E^1) + Q^2(E^2) = D_t T^t + D_x T^x$$

and

$$E_{(u,v)}[D_t T^t + D_x T^x] = 0.$$

In each case,  $T^t$  is the conserved density.

**Kerr law**

This law is also known as the cubic nonlinearity and is considered to be the simplest known form of nonlinearity. Most optical fibers that are commercially available nowadays obey this Kerr law of nonlinearity. Therefore, in this first section the attention will be on optical metamaterials with cubic nonlinearity. In this case

$$F(u) = bu$$

for some non-zero constant  $b$ . Therefore, the governing equation given by (1) with Kerr law nonlinearity reduces to [2,10]

$$iq_t + aq_{xx} + b|q|^2q = i\alpha q_x + i\lambda(|q|^2q)_x + iv(|q|^2q)_x + \theta_1(|q|^2q)_{xx} + \theta_2|q|^2q_{xx} + \theta_3q^2q_{xx}^* \tag{5}$$

In order to locate the conservation laws for Eq. (5), the complex valued function is split into real and imaginary parts as:

$$q(x, t) = u(x, t) + iv(x, t).$$

This leads to the decomposition of Eq. (5) as

$$\begin{aligned} & -\theta_2 v^2 u_{xx} - \theta_3 u^2 u_{xx} - 6\theta_1 uu_x^2 - 2\theta_1 uv_x^2 - \theta_2 u^2 u_{xx} - 3\theta_1 u^2 u_{xx} \\ & - \theta_1 v^2 u_{xx} + buv^2 + 2vv^2 v_x + \theta_3 v^2 u_{xx} + \lambda u^2 v_x + 3\lambda v^2 v_x \\ & - 4\theta_1 vu_x v_x - 2\theta_1 uvv_{xx} + 2vuvu_x - 2\theta_3 uvv_{xx} + 2\lambda uvu_x \\ & - v_t + bu^3 + \alpha v_x + au_{xx} = 0, -\alpha u_x + bv^3 + av_{xx} - \theta_2 u^2 v_{xx} \\ & - \theta_2 v^2 v_{xx} - 2\theta_1 vu_x^2 - 6\theta_1 vv_x^2 - \theta_1 u^2 v_{xx} - 3\theta_1 v^2 v_{xx} - \lambda v^2 u_x \\ & - 2vu^2 u_x + \theta_3 u^2 v_{xx} - 3\lambda u^2 u_x - \theta_3 v^2 v_{xx} + bu^2 v - 2\theta_3 uvu_{xx} \\ & - 2\theta_1 uvu_{xx} - 4\theta_1 uu_x v_x - 2\lambda uvv_x - 2vuvv_x + u_t = 0. \end{aligned} \tag{6}$$

The lengthy calculations for the system (6) lead to the following multipliers

$$\begin{aligned} Q^1 &= -c_1 v((\theta_1 + \theta_2 - \theta_3)u^2 + (\theta_1 + \theta_2 - \theta_3)v^2 - a)^{\frac{\theta_3 - \theta_2 + \theta_1}{\theta_1 + \theta_2 - \theta_3}} \\ Q^2 &= c_1 u((\theta_1 + \theta_2 - \theta_3)u^2 + (\theta_1 + \theta_2 - \theta_3)v^2 - a)^{\frac{\theta_3 - \theta_2 + \theta_1}{\theta_1 + \theta_2 - \theta_3}} \end{aligned} \tag{7}$$

For the multipliers (7), we can not obtain conservation laws of the system (6) because of only implemented for integer exponents.

**Case-1:**

If we set  $\theta_1 = \theta_2 = \theta_3 = 1$  in Eq. (5), for example, we have the following multipliers and corresponding conserved vectors

$$\begin{aligned} Q^1 &= v(-u^2 - v^2 - a), \\ Q^2 &= -u(-u^2 - v^2 + a), \end{aligned} \tag{8}$$

$$\begin{aligned} T^t &= -\frac{(u^2 + v^2)(-u^2 - v^2 + 2a)}{4}, \\ T^x &= \frac{(u^2 + v^2)(-u^2 - v^2 + 2a)\alpha}{4} \\ &+ \frac{(u^2 + v^2)^2(-2u^2 - 2v^2 + 3a)\lambda}{4} \\ &+ \frac{(u^2 + v^2)^2(-2u^2 - 2v^2 + 3a)v}{6} + v(-u^2 - v^2 + a)^2 u_x \\ &- u(-u^2 - v^2 + a)^2 v_x \end{aligned} \tag{9}$$

Thus, the conserved density for (5) is

$$\Phi^t = -\frac{|q|^2(-|q|^2 + 2a)}{4} \tag{10}$$

**Case-2:**

If we set  $\theta_1 = 3, \theta_2 = 1, \theta_3 = 2$  in Eq. (5), for example, we have the following multipliers and corresponding conserved vectors

$$\begin{aligned} Q^1 &= -v(-2u^2 - 2v^2 + a)^2, \\ Q^2 &= u(-2u^2 - 2v^2 + a)^2, \end{aligned} \tag{11}$$

$$\begin{aligned} T^t &= \frac{(u^2 + v^2)(4u^4 + 8u^2 v^2 + 4v^4 - 6au^2 - 6av^2 + 3a^2)}{6}, \\ T^x &= -\frac{(u^2 + v^2)(4u^4 + 8u^2 v^2 + 4v^4 - 6au^2 - 6av^2 + 3a^2)\alpha}{6} \\ &- \frac{(u^2 + v^2)^2(6u^4 + 12u^2 v^2 + 6v^4 - 8au^2 - 8av^2 + 3a^2)\lambda}{4} \\ &- \frac{(u^2 + v^2)^2(6u^4 + 12u^2 v^2 + 6v^4 - 8au^2 - 8av^2 + 3a^2)v}{6} \\ &- v(-2u^2 - 2v^2 + a)^3 u_x + u(-2u^2 - 2v^2 + a)^3 v_x \end{aligned} \tag{12}$$

and

$$\Phi^t = \frac{|q|^2(4|q|^4 - 6a|q|^2 + 3a^2)}{6}. \tag{13}$$

**Case-3:**

If we set  $\theta_1 = 2, \theta_2 = 1, \theta_3 = 2$  in Eq. (5), for example, we have the following multipliers and corresponding conserved vectors

$$\begin{aligned} Q^1 &= v(-u^2 - v^2 + a)^3, \\ Q^2 &= -u(-u^2 - v^2 + a)^3, \end{aligned} \tag{14}$$

---


$$\begin{aligned} T^t &= -\frac{(u^2 + v^2)(-u^2 - v^2 + 2a)(u^4 + 2u^2v^2 + v^4 - 2au^2 - 2av^2 + 2a^2)}{8}, \\ T^x &= \frac{(u^2 + v^2)(-u^2 - v^2 + 2a)(u^4 + 2u^2v^2 + v^4 - 2au^2 - 2av^2 + 2a^2)\alpha}{8} \\ &\quad + \frac{3(u^2 + v^2)^2(-4u^6 - 12u^4v^2 - 12u^2v^4 - 4v^6 + 15au^4 + 30au^2v^2 + 15av^4 - 20a^2u^2 - 20a^2v^2 + 10a^3)\lambda}{40} \\ &\quad + \frac{(u^2 + v^2)^2(-4u^6 - 12u^4v^2 - 12u^2v^4 - 4v^6 + 15au^4 + 30au^2v^2 + 15av^4 - 20a^2u^2 - 20a^2v^2 + 10a^3)v}{20} + v(-u^2 - v^2 + a)^4u_x - u(-u^2 - v^2 + a)^4v_x \end{aligned} \tag{15}$$


---

and

$$\Phi^t = -\frac{|q|^2(-|q|^2 + 2a)(|q|^4 - 2a|q|^2 + 2a^2)}{8}. \tag{16}$$

**Case-4:**

If we set  $\theta_1 = 5, \theta_2 = 1, \theta_3 = 4$  in Eq. (5), for example, we have the following multipliers and corresponding conserved vectors

$$\begin{aligned} Q^1 &= -v(-2u^2 - 2v^2 + a)^4, \\ Q^2 &= u(-2u^2 - 2v^2 + a)^4, \end{aligned} \tag{17}$$

---


$$\begin{aligned} T^t &= \frac{(u^2 + v^2)(16(u^2 + v^2)^4 - 40a(u^2 + v^2)^3 + 40a^2(u^2 + v^2)^2 - 20a^3(u^2 + v^2) + 5a^4)}{10}, \\ T^x &= -\frac{(u^2 + v^2)(16(u^2 + v^2)^4 - 40a(u^2 + v^2)^3 + 40a^2(u^2 + v^2)^2 - 20a^3(u^2 + v^2) + 5a^4)\alpha}{10} \\ &\quad - \frac{(u^2 + v^2)^2(80(u^2 + v^2)^4 - 192a(u^2 + v^2)^3 + 180a^2(u^2 + v^2)^2 - 80a^3(u^2 + v^2) + 15a^4)\lambda}{20} \\ &\quad - \frac{(u^2 + v^2)^2(80(u^2 + v^2)^4 - 192a(u^2 + v^2)^3 + 180a^2(u^2 + v^2)^2 - 80a^3(u^2 + v^2) + 15a^4)v}{30} - v(-2u^2 - 2v^2 + a)^5u_x + u(-2u^2 - 2v^2 + a)^5v_x \end{aligned} \tag{18}$$


---

and

$$\Phi^t = \frac{|q|^2(16|q|^8 - 40a|q|^6 + 40a^2|q|^4 - 20a^3|q|^2 + 5a^4)}{10}. \tag{19}$$

Thus, we have infinitely many conservation laws of the system (6).

The 1-soliton solution to the perturbed NLSE (5) is now given by [2,10]

$$q(x, t) = A \operatorname{sech}[B(x - ct)]e^{i(-\kappa x + \omega t + \theta_0)}, \tag{20}$$

where  $A$  is the amplitude of the soliton,  $B$  is the inverse width of the soliton. From the phase component,  $\kappa$  is the soliton frequency,  $\omega$  is the wave number of the soliton and  $\theta_0$  is the phase constant. The speed of the soliton is given by

$$c = -\alpha - 2a\kappa. \tag{21}$$

For the four cases discussed above, the conserved quantities are now given by:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} (2a|q|^2 - |q|^4) dx \\ &= A^2 \int_{-\infty}^{\infty} (2a \operatorname{sech}^2 \tau - A^2 \operatorname{sech}^4 \tau) dx = \frac{4A^2}{3B} (3a - A^2), \end{aligned} \tag{22}$$

where the notation

$$\tau = B(x - ct) \tag{23}$$

is adopted here.

---

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} (3a^2|q|^2 - 6a|q|^4 + 4|q|^6) dx \\ &= A^2 \int_{-\infty}^{\infty} (3a^2 \operatorname{sech}^2 \tau - 6aA^2 \operatorname{sech}^4 \tau + 4A^4 \operatorname{sech}^6 \tau) dx \\ &= \frac{2A^2}{15B} (45a^2 - 60aA^2 + 32A^4) \end{aligned} \tag{24}$$

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} |q|^2(2a - |q|^2)(2a^2 - 2a|q|^2 + |q|^4) dx \\ &= \int_{-\infty}^{\infty} (4a^3|q|^2 - 6a^2|q|^4 + 4a|q|^6 - |q|^8) dx \\ &= A^2 \int_{-\infty}^{\infty} (4a^3 \operatorname{sech}^2 \tau - 6a^2A^2 \operatorname{sech}^4 \tau + 4aA^4 \operatorname{sech}^6 \tau - A^6 \operatorname{sech}^8 \tau) dx \\ &= \frac{8A^2}{35B} (35a^3 - 35a^2A^2 + 280aA^4 - 4A^6) \end{aligned} \tag{25}$$

$$\begin{aligned}
 I_4 &= \int_{-\infty}^{\infty} (5a^4|q|^2 - 20a^3|q|^4 + 40a^2|q|^6 - 40a|q|^8 + 16|q|^{10}) dx \\
 &= A^2 \int_{-\infty}^{\infty} (5a^4 \operatorname{sech}^2 \tau - 20a^3 A^2 \operatorname{sech}^4 \tau + 40a^2 A^4 \operatorname{sech}^6 \tau \\
 &\quad - 40a A^6 \operatorname{sech}^8 \tau + 16A^8 \operatorname{sech}^{10} \tau) dx = \frac{2A^2}{315B} (1575a^4 - 12600a^3 A^2 \\
 &\quad + 6720a^2 A^4 - 5760a A^6 + 2048A^8) \tag{26}
 \end{aligned}$$

**Parabolic law**

This law is alternatively known as the cubic–quintic nonlinearity and is studied in nonlinear interaction between Langmuir waves and electrons. It describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces. It takes the form

$$F(u) = b_1 u + b_2 u^2$$

for non-zero constants  $b_1$  and  $b_2$ . For parabolic law medium, the NLSE given by (1) changes to [10]

$$\begin{aligned}
 i q_t + a q_{xx} + (b_1 |q|^2 + b_2 |q|^4) q &= i x q_x + i \lambda (|q|^2 q)_x \\
 + i v (|q|^2)_x + \theta_1 (|q|^2 q)_{xx} &+ \theta_2 |q|^2 q_{xx} + \theta_3 q^2 q_{xx}^* \tag{27}
 \end{aligned}$$

Eq. (27) splits as

$$\begin{aligned}
 \theta_3 u_{xx} v^2 - 3\theta_1 u_{xx} u^2 - \theta_1 u_{xx} v^2 - \theta_2 u_{xx} u^2 - \theta_2 u_{xx} v^2 - \theta_3 u_{xx} u^2 \\
 + a u_{xx} + b_2 u^5 + b_1 u^3 + \alpha v_x - v_t + 2\lambda u v u_x + 2\nu u v u_x - 4\theta_1 v u_x v_x \\
 - 2\theta_3 v_{xx} u v - 2\theta_1 v_{xx} u v + \lambda u^2 v_x + 3\lambda v^2 v_x + 2\nu v^2 v_x + 2b_2 u^3 v^2 \\
 + b_2 u v^4 + b_1 u v^2 - 6\theta_1 u u_x^2 - 2\theta_1 u v_x^2 = 0, \\
 -\alpha u_x + b_1 v^3 + a v_{xx} + b_2 v^5 - \theta_2 u^2 v_{xx} - \theta_2 v^2 v_{xx} - 6\theta_1 v v_x^2 \\
 - 3\theta_1 v^2 v_{xx} - 2\theta_1 v u_x^2 - \lambda v^2 u_x - 2\nu u^2 u_x - \theta_1 u^2 v_{xx} + 2b_2 u^2 v^3 \\
 - 3\lambda u^2 u_x - \theta_3 v^2 v_{xx} + \theta_3 u^2 v_{xx} + b_1 u^2 v + b_2 u^4 v - 2\theta_3 u v u_{xx} \\
 - 4\theta_1 u u_x v_x - 2\nu u v v_x - 2\theta_1 u v u_{xx} - 2\lambda u v v_x + u_t = 0 \tag{28}
 \end{aligned}$$

For the system (28), we obtain Eqs. (7)–(19).

For parabolic law of nonlinearity, the bright 1-soliton solution in optical metamaterials is given by [10]:

$$q(x, t) = \frac{A}{\sqrt{D + \cosh[B(x - ct)]}} e^{i(-kx + \omega t + \theta_0)} \tag{29}$$

where  $D$  is a newly introduced parameter whose restriction will be given later. The remaining parameters have the same interpretations as Kerr law soliton. The conservation laws for parabolic law nonlinearity are now given by

$$\begin{aligned}
 I_1 &= \int_{-\infty}^{\infty} (2a|q|^2 - |q|^4) dx \\
 &= A^2 \int_{-\infty}^{\infty} \left\{ \frac{2a}{D + \cosh \tau} - \frac{A^2}{(D + \cosh \tau)^2} \right\} dx \\
 &= \frac{2A^2}{3B} \left\{ 6aF\left(1, 1; \frac{3}{2}; \frac{1-D}{2}\right) - A^2 F\left(2, 2; \frac{5}{2}; \frac{1-D}{2}\right) \right\} \tag{30}
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_{-\infty}^{\infty} (3a^2|q|^2 - 6a|q|^4 + 4|q|^6) dx \\
 &= A^2 \int_{-\infty}^{\infty} \left\{ \frac{3a^2}{D + \cosh \tau} - \frac{6aA^2}{(D + \cosh \tau)^2} + \frac{4A^4}{(D + \cosh \tau)^3} \right\} dx \\
 &= \frac{2A^2}{15B} \left\{ 45a^2 F\left(1, 1; \frac{3}{2}; \frac{1-D}{2}\right) - 30aA^2 F\left(2, 2; \frac{5}{2}; \frac{1-D}{2}\right) \right. \\
 &\quad \left. + 8A^4 F\left(3, 3; \frac{7}{2}; \frac{1-D}{2}\right) \right\} \tag{31}
 \end{aligned}$$

$$\begin{aligned}
 I_3 &= \int_{-\infty}^{\infty} |q|^2 (2a - |q|^2) (2a^2 - 2a|q|^2 + |q|^4) dx \\
 &= \int_{-\infty}^{\infty} (4a^3|q|^2 - 6a^2|q|^4 + 4a|q|^6 - |q|^8) dx \\
 &= A^2 \int_{-\infty}^{\infty} \left\{ \frac{4a^3}{D + \cosh \tau} - \frac{6a^2A^2}{(D + \cosh \tau)^2} + \frac{4aA^4}{(D + \cosh \tau)^3} \right. \\
 &\quad \left. - \frac{A^6}{(D + \cosh \tau)^4} \right\} dx = \frac{4A^2}{105B} \left\{ 210a^3 F\left(1, 1; \frac{3}{2}; \frac{1-D}{2}\right) \right. \\
 &\quad - 105a^2 A^2 F\left(2, 2; \frac{5}{2}; \frac{1-D}{2}\right) + 28aA^4 F\left(3, 3; \frac{7}{2}; \frac{1-D}{2}\right) \\
 &\quad \left. - 3A^6 F\left(4, 4; \frac{9}{2}; \frac{1-D}{2}\right) \right\} \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 I_4 &= \int_{-\infty}^{\infty} (5a^4|q|^2 - 20a^3|q|^4 + 40a^2|q|^6 - 40a|q|^8 + 16|q|^{10}) dx \\
 &= A^2 \int_{-\infty}^{\infty} \left\{ \frac{5a^4}{D + \cosh \tau} - \frac{20a^3A^2}{(D + \cosh \tau)^2} + \frac{40a^2A^4}{(D + \cosh \tau)^3} \right. \\
 &\quad \left. - \frac{40aA^6}{(D + \cosh \tau)^4} + \frac{16A^8}{(D + \cosh \tau)^5} \right\} dx \\
 &= \frac{2A^2}{315B} \left\{ 1575a^4 F\left(1, 1; \frac{3}{2}; \frac{1-D}{2}\right) - 2100a^3 A^2 F\left(2, 2; \frac{5}{2}; \frac{1-D}{2}\right) \right. \\
 &\quad + 1680a^2 A^4 F\left(3, 3; \frac{7}{2}; \frac{1-D}{2}\right) - 720aA^6 F\left(4, 4; \frac{9}{2}; \frac{1-D}{2}\right) \\
 &\quad \left. + 128A^8 F\left(5, 5; \frac{11}{2}; \frac{1-D}{2}\right) \right\} \tag{33}
 \end{aligned}$$

Here, in (30)–(33) Gauss' hypergeometric function is defined as:

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n n!} \tag{34}$$

and the Pochhammer symbol is

$$(p)_n = \begin{cases} 1 & n = 0, \\ p(p+1) \cdots (p+n-1) & n > 0. \end{cases} \tag{35}$$

The convergence criteria for hypergeometric function is

$$|z| < 1. \tag{36}$$

which, for (30)–(33), implies

$$-1 < D < 3. \tag{37}$$

Furthermore, Rabbe's criteria of convergence implies

$$\gamma < \alpha + \beta, \tag{38}$$

which is valid for all of the hypergeometric functions listed above.

**Conclusions**

This paper retrieved conservation laws for solitons in optical metamaterials by the aid of Lie symmetry analysis. There are two forms of nonlinear media that are studied in this paper, which are Kerr law and parabolic law. Three of the important laws are not included in this paper since it is not possible to retrieve conserved densities for power law, dual-power law and logarithmic law by Lie symmetry. However, this paper is encouraging to seek for such densities for various other laws in the context of optical metamaterials. The results with those additional laws will be reported in future.

## Acknowledgements

The third author (QZ) was funded by the National Science Foundation of Hubei Province in China under the Grant No. 2015CFC891. The fourth author (SPM) would like to thank the research support provided by the Department of Mathematics and Statistics at Tshwane University of Technology and the support from the South African National Foundation under Grant No. 92052 IRF1202210126. The research work of sixth author (MB) was supported by Qatar National Research Fund (QNRF) under the Grant No. NPRP 8-028-1-001. The authors also declare that there is no conflict of interest.

## References

- [1] Assouar B, Oudich M, Zhou X. Acoustic metamaterials for sound mitigation'. *CR Phys* 2016;17(5):524–32.
- [2] Biswas A, Khan KR, Mahmood MF, Belic M. Bright and dark solitons in optical metamaterials. *Optik* 2014;125(13):3299–302.
- [3] Douvagai, Salathiel Y, Betchewe G, Doka SY, Crepin KT. Electromagnetic wave solitons in optical metamaterials. *Optik* 2017;140:735–42.
- [4] Erfaninia H, Rostami A. Group velocity reduction in multilayer metamaterial waveguide. *Optik* 2013;124(12):1230–3.
- [5] Kara AH, Biswas A, Belic M. Conservation laws for optical solitons in birefringent fibers and magneto-optic waveguides. *Optik* 2016;127(24):11662–73.
- [6] Kinsler P, McCall MW. The futures of transformations and metamaterial. *Photon Nanostruct – Fundam Appl* 2015;15:10–23.
- [7] Lima J, Adam J, Rego D, Esquerre V, Bordo V. Optical properties of nanowire metamaterials with gain. *Opt Commun* 2016;379:25–31.
- [8] Mulla B, Sabah C. Multi-band metamaterial absorber topology for infrared frequency regime. *Physica E* 2017;86:44–51.
- [9] Pranav US, Sudheesh S, Stanly P, Sankar S, Pradeep A. Metamaterial based energy harvester'. *Proc Comput Sci* 2016;93:74–80.
- [10] Xu Y, Savescu M, Khan KR, Mahmood MF, Biswas A, Belic M. Soliton propagation through nanoscale waveguides in optical metamaterials'. *Opt Laser Technol* 2016;77:177–86.