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Item Type	Article
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DOI	<a href="https://doi.org/10.5269/bspm.v27i2.10205">https://doi.org/10.5269/bspm.v27i2.10205</a>
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Download date	2026-05-09 17:29:26
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Link to Item	<a href="https://hdl.handle.net/20.500.14519/1105">https://hdl.handle.net/20.500.14519/1105</a>



## On $(\delta, p)$ -continuous functions and $(\delta, p)$ -closed graphs

M. Caldas, E. Ekici, S. Jafari and S. P. Moshokoa

**ABSTRACT:** It is the object of this paper to introduce the notions of  $(\delta, p)$ -continuity and  $(\delta, p)$ -closed graphs by utilizing the notion of  $(\delta, p)$ -open sets and investigate the fundamental properties of  $(\delta, p)$ -continuous functions and also present some properties of functions with  $(\delta, p)$ -closed graphs.

**Key Words:** Topological spaces,  $(\delta, p)$ -open set,  $(\delta, p)$ -closed graph,  $(\delta, p)$ - $T_1$ ,  $(\delta, p)$ -continuous.  $(\delta, p)$ - $W$ -continuous.

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### 1. Introduction

In this paper  $X$  and  $Y$  denote the topological spaces. Let  $A$  be a subset of  $X$ . We denote the interior and the closure of a set  $A$  by  $Int(A)$  and  $Cl(A)$  respectively. Jafari [2] introduced the notion of pre-regular  $p$ -open sets and further investigated its fundamental properties in [3]. A subset  $A$  of a topological space  $(X, \tau)$  is called a *pre-regular  $p$ -open* [2] if  $A = pInt(pCl(A))$ . Now we recall the following notions from [1] which will be used in the sequel: A point  $x \in X$  is called the  $(\delta, p)$ -cluster point of  $A$  if  $A \cap U \neq \emptyset$  for every pre-regular  $p$ -open set  $U$  of  $X$  containing  $x$ . The set of all  $(\delta, p)$ -cluster points of  $A$  is called the  $(\delta, p)$ -closure of  $A$ , denoted by  $\delta Cl_p(A)$ . If  $\delta Cl_p(A) = A$ , then  $A$  is called  $(\delta, p)$ -closed. The complement of a  $(\delta, p)$ -closed set is called  $(\delta, p)$ -open. We say that a set  $U$  in a topological space  $(X, \tau)$  is a  $(\delta, p)$ -neighborhood of a point  $x$  if  $U$  contains a  $(\delta, p)$ -open set to which  $x$  belongs. We denote the collection of all  $(\delta, p)$ -open (respectively  $(\delta, p)$ -closed) sets by  $\delta PO(X, \tau)$  (respectively  $\delta PC(X, \tau)$ ).

In this paper we offer a new class of functions called  $(\delta, p)$ -continuous functions and a new notion of the graph of a function called a  $(\delta, p)$ -closed graph. We also investigate some of their fundamental properties.

### 2. Some properties

**Definition 2.1** A function  $f : X \rightarrow Y$  is said to be  $(\delta, p)$ -continuous if for every open set  $V$  of  $Y$ ,  $f^{-1}(V)$  is  $(\delta, p)$ -open in  $X$ .

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2000 Mathematics Subject Classification: 54C10, 54D10

**Theorem 2.1** *The following are equivalent for a function  $f : X \rightarrow Y$ :*

- (1)  $f$  is  $(\delta, p)$ -continuous,
- (2) The inverse image of every closed set in  $Y$  is  $(\delta, p)$ -closed in  $X$ ,
- (3) For each subset  $A$  of  $X$ ,  $f(\delta Cl_p(A)) \subset Cl(f(A))$ ,
- (4) For each subset  $B$  of  $Y$ ,  $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$ .

*Proof.* (1)  $\Leftrightarrow$  (2) : Obvious.

(3)  $\Leftrightarrow$  (4) : Let  $B$  is any subset of  $Y$ . Then by (3), we have  $f(\delta Cl_p(f^{-1}(B))) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$ . This implies  $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(f(\delta Cl_p(f^{-1}(B)))) \subset f^{-1}(Cl(B))$ .

Conversely, let  $B = f(A)$  where  $A$  is a subset of  $X$ . Then, by (4), we have,  $\delta Cl_p(A) \subset \delta Cl_p(f^{-1}(f(A))) \subset f^{-1}(Cl(f(A)))$ . Thus,  $f(\delta Cl_p(A)) \subset Cl(f(A))$ .

(2)  $\Rightarrow$  (4) : Let  $B \subset Y$ . Since  $f^{-1}(Cl(B))$  is  $(\delta, p)$ -closed and  $f^{-1}(B) \subset f^{-1}(Cl(B))$ , then  $\delta Cl_p(f^{-1}(B)) \subset f^{-1}(Cl(B))$ .

(4)  $\Rightarrow$  (2) : Let  $K \subset Y$  be a closed set. By (4),  $\delta Cl_p(f^{-1}(K)) \subset f^{-1}(Cl(K)) = f^{-1}(K)$ . Thus,  $f^{-1}(K)$  is  $(\delta, p)$ -closed.

Recall that for a function  $f : X \rightarrow Y$ , the subset  $\{(x, f(x)) \mid x \in X\}$  of the product space  $X \times Y$  is called the graph of  $f$  and is denoted by  $G(f)$ .

**Definition 2.2** *For a function  $f : X \rightarrow Y$ , the graph  $G(f) = \{(x, f(x)) \mid x \in X\}$  is said to be  $(\delta, p)$ -closed if for each  $(x, y) \in X \times Y \setminus G(f)$ , there exist  $U \in \delta PO(X, x)$  and an open set  $V$  of  $Y$  containing  $y$  such that  $(U \times V) \cap G(f) = \emptyset$ .*

**Lemma 2.1** *Let  $f : X \rightarrow Y$  be a function. Then the graph  $G(f)$  is  $(\delta, p)$ -closed in  $X \times Y$  if and only if for each point  $(x, y) \in X \times Y \setminus G(f)$ , there exist a  $(\delta, p)$ -open set  $U$  and an open set  $V$  containing  $x$  and  $y$ , respectively, such that  $f(U) \cap V = \emptyset$ .*

*Proof.* It follows readily from the above definition.

**Definition 2.3** *A space  $X$  is said to be  $(\delta, p)$ - $T_1$  [1] if for each pair of distinct points  $x$  and  $y$  of  $X$ , there exist a  $(\delta, p)$ -open set  $U$  containing  $x$  but not  $y$  and a  $(\delta, p)$ -open set  $V$  containing  $y$  but not  $x$ .*

**Theorem 2.2** *If  $f : X \rightarrow Y$  is an injective function with the  $(\delta, p)$ -closed graph, then  $X$  is  $(\delta, p)$ - $T_1$ .*

*Proof.* Let  $x$  and  $y$  be two distinct points of  $X$ . Then  $f(x) \neq f(y)$ . Thus there exist a  $(\delta, p)$ -open set  $U$  and an open set  $V$  containing  $x$  and  $f(y)$ , respectively, such that  $f(U) \cap V = \emptyset$ . Therefore  $y \notin U$  and it follows that  $X$  is  $(\delta, p)$ - $T_1$ .

Recall that a space  $X$  is said to be  $T_1$  if for each pair of distinct points  $x$  and  $y$  of  $X$ , there exist an open set  $U$  containing  $x$  but not  $y$  and an open set  $V$  containing  $y$  but not  $x$ .

**Theorem 2.3** *If  $f : X \rightarrow Y$  is a surjective function with the  $(\delta, p)$ -closed graph, then  $Y$  is  $T_1$ .*

*Proof.* Let  $y_1$  and  $y_2$  be two distinct points of  $Y$ . Since  $f$  is surjective, there exists a point  $x$  in  $X$  such that  $f(x) = y_2$ . Therefore  $(x, y_1) \notin G(f)$ . By Lemma 2.1, there exist a  $(\delta, p)$ -open set  $U$  and an open set  $V$  containing  $x$  and  $y_1$ , respectively, such that  $f(U) \cap V = \emptyset$ . It follows that  $y_2 \notin V$ . Hence  $Y$  is  $T_1$ .

**Definition 2.4** A function  $f : X \rightarrow Y$  is said to be  $(\delta, p)$ - $W$ -continuous if for each  $x \in X$  and each open set  $V$  of  $Y$  containing  $f(x)$ , there exists a  $(\delta, p)$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset Cl(V)$ .

**Theorem 2.4** If  $f : X \rightarrow Y$  is  $(\delta, p)$ - $W$ -continuous and  $Y$  is Hausdorff, then  $G(f)$  is  $(\delta, p)$ -closed.

*Proof.* Suppose that  $(x, y) \notin G(f)$ , then  $f(x) \neq y$ . By the fact that  $Y$  is Hausdorff, there exist open sets  $W$  and  $V$  such that  $f(x) \in W$ ,  $y \in V$  and  $V \cap W = \emptyset$ . It follows that  $Cl(W) \cap V = \emptyset$ . Since  $f$  is  $(\delta, p)$ - $W$ -continuous, there exists  $U \in \delta PO(X, x)$  such that  $f(U) \subset Cl(W)$ . Hence, we have  $f(U) \cap V = \emptyset$ . This means that  $G(f)$  is  $(\delta, p)$ -closed.

**Corollary 2.4A** If  $f : X \rightarrow Y$  is  $(\delta, p)$ -continuous and  $Y$  is Hausdorff, then  $G(f)$  is  $(\delta, p)$ -closed in  $X \times Y$ .

**Definition 2.5** A subset  $A$  of a space  $X$  is said to be  $(\delta, p)$ -compact relative to  $X$  if every cover of  $A$  by  $(\delta, p)$ -open sets of  $X$  has a finite subcover.

**Theorem 2.5** Let  $f : X \rightarrow Y$  have a  $(\delta, p)$ -closed graph. If  $K$  is  $(\delta, p)$ -compact relative to  $X$ , then  $f(K)$  is closed in  $Y$ .

*Proof.* Suppose  $y \notin f(K)$ . For each  $x \in K$ ,  $f(x) \neq y$ . By Lemma 2.1, there exist  $U_x \in \delta PO(X, x)$  and an open neighbourhood  $V_x$  of  $y$  such that  $f(U_x) \cap V_x = \emptyset$ . The family  $\{U_x \mid x \in K\}$  is a cover of  $K$  by  $(\delta, p)$ -open sets of  $X$  and there exists a finite subset  $K_0$  of  $K$  such that  $K \subset \bigcup\{U_x \mid x \in K_0\}$ . Put  $V = \bigcap\{V_x \mid x \in K_0\}$ . Then  $V$  is an open neighbourhood of  $y$  and  $f(K) \cap V = \emptyset$ . This means that  $f(K)$  is closed in  $Y$ .

**Definition 2.6** A function  $f : X \rightarrow Y$  is called perfectly continuous [4] if for each open set  $A \subset Y$ ,  $f^{-1}(A)$  is open and closed in  $X$ .

**Lemma 2.2** ([3]) If  $A$  and  $B$  are pre-regular  $p$ -open sets of the spaces  $X$  and  $Y$ , respectively, then  $A \times B$  is a pre-regular  $p$ -open set of  $X \times Y$ .

**Theorem 2.6** If  $f : X \rightarrow Z$  has a  $(\delta, p)$ -closed graph  $G(f)$  and  $g : Y \rightarrow Z$  is a perfectly continuous function, then the set  $\{(x, y) : f(x) = g(y)\}$  is  $(\delta, p)$ -closed in  $X \times Y$ .

*Proof.* Let  $A = \{(x, y) : f(x) = g(y)\}$  and  $(x, y) \in X \setminus A$ . We have  $f(x) \neq g(y)$  and then  $(x, g(y)) \in (X \times Z) \setminus G(f)$ . Since  $f$  has a  $(\delta, p)$ -closed graph  $G(f)$ , there exist a  $(\delta, p)$ -open set  $U$  and an open set  $V$  containing  $x$  and  $g(y)$ , respectively

such that  $f(U) \cap V = \emptyset$ . This implies that there exists a pre-regular  $p$ -open set  $N$  containing  $x$  such that  $N \subset U$  and  $f(N) \cap V = \emptyset$ . Since  $g$  is a perfectly continuous function, then there exist an open and closed set  $G$  containing  $y$  such that  $g(G) \subset V$ . We have  $f(U) \cap g(G) = \emptyset$ . This implies that  $(N \times G) \cap A = \emptyset$ . Since  $N \times G$  is pre-regular  $p$ -open, then  $(x, y) \notin \delta Cl_p(A)$ . Thus,  $E$  is  $(\delta, p)$ -closed in  $X \times Y$ .

**Corollary 2.6B** *If  $f : X \rightarrow Z$  is a  $(\delta, p)$ -continuous function and  $g : Y \rightarrow Z$  is a perfectly continuous function and  $Z$  is Hausdorff, then the set  $\{(x, y) : f(x) = g(y)\}$  is  $(\delta, p)$ -closed in  $X \times Y$ .*

*Proof.* It follows from Corollary 2.6A and Theorem 2.6.

**Theorem 2.7** *If  $f : X \rightarrow Y$  is a  $(\delta, p)$ -continuous function and  $Y$  is Hausdorff, then the set  $\{(x, y) \in X \times X : f(x) = f(y)\}$  is  $(\delta, p)$ -closed in  $X \times X$ .*

*Proof.* Let  $A = \{(x, y) : f(x) = f(y)\}$  and let  $(x, y) \in (X \times X) \setminus A$ . It follows that  $f(x) \neq f(y)$ . Since  $Y$  is Hausdorff, there exist open sets  $U$  and  $V$  containing  $f(x)$  and  $f(y)$ , respectively, such that  $U \cap V = \emptyset$ . Since  $f$  is  $(\delta, p)$ -continuous, there exist pre-regular  $p$ -open sets  $H$  and  $G$  in  $X$  containing  $x$  and  $y$ , respectively, such that  $f(H) \subset U$  and  $f(G) \subset V$ . This implies  $(H \times G) \cap A = \emptyset$ . We have  $H \times G$  is a pre-regular  $p$ -open set in  $X \times X$  containing  $(x, y)$ . Hence,  $A$  is  $(\delta, p)$ -closed in  $X \times X$ .

**Definition 2.7** *A function  $f : X \rightarrow Y$  is called contra  $(\delta, p)$ -open if the image of every  $(\delta, p)$ -open set in  $X$  is closed in  $Y$ .*

**Theorem 2.8** *If  $f : X \rightarrow Y$  is a contra  $(\delta, p)$ -open function such that inverse image of each point of  $Y$  is  $(\delta, p)$ -closed, then  $f$  has a  $(\delta, p)$ -closed graph  $G(f)$ .*

*Proof.* Let  $(x, y) \in X \setminus G(f)$ . We have  $x \notin f^{-1}(y)$ . Since  $f^{-1}(y)$  is  $(\delta, p)$ -closed, there exists a pre-regular  $p$ -open set  $A$  containing  $x$  such that  $A \cap f^{-1}(y) = \emptyset$ . Since  $f$  is contra  $(\delta, p)$ -open, then  $f(A)$  is closed. This implies that there exist an open set  $B$  in  $Y$  containing  $y$  such that  $f(A) \cap B = \emptyset$ . Hence,  $f$  has a  $(\delta, p)$ -closed graph  $G(f)$ .

**Theorem 2.9** *If  $f : (X, \tau) \rightarrow (Y, \sigma)$  has a  $(\delta, p)$ -closed graph  $G(f)$ , then for each  $x \in X$ ,  $\{f(x)\} = \bigcap_{x \in A \in \delta PO(X, \tau)} Cl(f(A))$ .*

*Proof.* Suppose that  $y \neq f(x)$  and  $y \in \bigcap_{x \in A \in \delta PO(X, \tau)} Cl(f(A))$ . Then  $y \in Cl(f(A))$  for each  $x \in A \in \delta PO(X, \tau)$ . This implies that for each open set  $B$  containing  $y$ ,  $B \cap f(A) \neq \emptyset$ . Since  $(x, y) \notin G(f)$  and  $G(f)$  is a  $(\delta, p)$ -closed graph, this is a contradiction.

**Definition 2.8** *A space  $X$  is said to be  $(\delta, p)$ - $T_2$  if for each pair of distinct points  $x$  and  $y$  in  $X$ , there exist disjoint  $(\delta, p)$ -open sets  $A$  and  $B$  in  $X$  such that  $x \in A$  and  $y \in B$ .*

**Definition 2.9** A function  $f : X \rightarrow Y$  is called  $(\delta, p)$ -open if the image of every  $(\delta, p)$ -open set in  $X$  is open in  $Y$ .

**Theorem 2.10** If  $f : X \rightarrow Y$  is a surjective  $(\delta, p)$ -open function with a  $(\delta, p)$ -closed graph  $G(f)$ , then  $Y$  is  $T_2$ .

*Proof.* Let  $y_1$  and  $y_2$  be any distinct points of  $Y$ . Since  $f$  is surjective  $f(x) = y_1$  for some  $x \in X$  and  $(x, y_2) \in (X \times Y) \setminus G(f)$ . This implies that there exist a  $(\delta, p)$ -open set  $A$  of  $X$  and an open set  $B$  of  $Y$  such that  $(x, y_2) \in A \times B$  and  $(A \times B) \cap G(f) = \emptyset$ . We have  $f(A) \cap B = \emptyset$ . Since  $f$  is  $(\delta, p)$ -open, then  $f(A)$  is open such that  $f(x) = y_1 \in f(A)$ . Thus,  $Y$  is  $T_2$ .

**Theorem 2.11** If  $f : X \rightarrow Y$  is a  $(\delta, p)$ -continuous injection and  $Y$  is  $T_2$ , then  $X$  is  $(\delta, p)$ - $T_2$ .

*Proof.* Let  $x$  and  $y$  in  $X$  be any pair of distinct points. Then there exist disjoint open sets  $A$  and  $B$  in  $Y$  such that  $f(x) \in A$  and  $f(y) \in B$ . Since  $f$  is  $(\delta, p)$ -continuous,  $f^{-1}(A)$  and  $f^{-1}(B)$  is  $(\delta, p)$ -open in  $X$  containing  $x$  and  $y$  respectively. We have  $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ . Thus,  $X$  is  $(\delta, p)$ - $T_2$ .

**Lemma 2.3** ([3]) If a space  $X$  is submaximal, then any finite intersection of pre-regular  $p$ -open sets is pre-regular  $p$ -open.

**Theorem 2.12** If  $f, g : X \rightarrow Y$  are  $(\delta, p)$ -continuous functions,  $X$  is submaximal and  $Y$  is Hausdorff, then the set  $\{x \in X : f(x) = g(x)\}$  is  $(\delta, p)$ -closed in  $X$ .

*Proof.* Let  $A = \{x \in X : f(x) = g(x)\}$ . Take  $x \in X \setminus A$ . We have  $f(x) \neq g(x)$ . Since  $Y$  is Hausdorff, then there exist open sets  $U$  and  $V$  in  $Y$  containing  $f(x)$  and  $g(x)$ , respectively, such that  $U \cap V = \emptyset$ . Since  $f$  and  $g$  are  $(\delta, p)$ -continuous, then  $f^{-1}(U)$  and  $g^{-1}(V)$  are  $(\delta, p)$ -open in  $X$  with  $x \in f^{-1}(U)$  and  $x \in g^{-1}(V)$ . Then there exist pre-regular  $p$ -open sets  $G$  and  $H$  such that  $x \in G \subset f^{-1}(U)$  and  $x \in H \subset g^{-1}(V)$ . Take  $K = G \cap H$ . By Lemma 2.3,  $K$  is pre-regular  $p$ -open. Thus,  $f(K) \cap g(K) = \emptyset$  and hence  $x \notin \delta Cl_p(A)$ . This shows that  $A$  is  $(\delta, p)$ -closed in  $X$ .

### Acknowledgments

S. P. Moshokoa acknowledges the support by the South African National Research Foundation under Grant number 2053847. Parts of this paper were written while S. Jafari was a visitor to the Department of Mathematical Sciences, University of South Africa during October 2005.

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