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Primitive shifts on ψ-spaces

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ABSTRACT

Let D be a countable discrete space and let $p \in D$. For any primitive shift $\sigma : D \to D \setminus \{p\}$ there are $2^{\mathfrak{c}}$ σ-invariant maximal almost disjoint families on D. This implies that there are $2^{\mathfrak{c}}$ pairwise non-homeomorphic Ψ^* spaces admitting primitive shifts. Under $\mathfrak{a} = \mathfrak{c}$ there exists a maximal almost disjoint family $\mathcal{F}$ on a countable discrete space D such that the spaces $\Psi(\mathcal{F})$ and $\Psi^*(\mathcal{F})$ admit no primitive shift.

1. Introduction

R.M. Crownover [1] was the first to give a basis free definition of a shift operator on a general Banach space. Holub [7] considered isometric shift operators.

Definition 1.1. A linear operator T on a Banach space E is called a shift operator if

(i) T is an isometry;
(ii) the range $T(E)$ has co-dimension 1;
(iii) $\bigcap_{n=1}^{\infty} T^n(E) = \{0\}$.

In [4] it is shown that a shift operator on a Banach space $C(X)$ of continuous (real- or complex-valued) functions on a compact space X can be represented in terms of functions on X. When the compact space X has a dense countable set of isolated points, $C(X)$ may not have a shift (see page 12 of [8]) or can have a rather simple representation as in the theorem below (see Theorem 2.4 of [4]).

Theorem 1.2. *Let p be an isolated point of a compact Hausdorff space X and $\sigma : X \to X \setminus \{p\}$ be a homeomorphism. If the orbit $\{\sigma^{-n}(p)\colon n = 0, 1, 2, \ldots\}$ is dense in X then the operator $T : C(X) \to C(X)$ assigning to each function $f \in C(X)$ the function $T(f)$ defined by*

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$$Tf(x) = f\big(\sigma(x)\big) \quad \text{for } x \in X \text{ and } x \neq p,$$
$$Tf(p) = 0$$

is a shift operator on a Banach space $C(X)$.

Definition 1.3. A closed embedding $\sigma : X \to X$ of a topological space X such that $X \setminus \sigma(X) = \{p\}$ is a singleton whose orbit $\{\sigma^{-n}(p)\colon\ n = 0, 1, 2, \ldots\}$ is dense in X is called a primitive shift on X.

Primitive shift is defined for compact spaces X in a different but equivalent way in [6] and [5].

Note that the point p is isolated, the orbit $\{\sigma^{-n}(p)\colon\ n = 0, 1, 2, \ldots\}$ of p is the set of all isolated points of X, σ is uniquely determined by its values on the set D of all isolated points of X, and the restriction of σ to the set D is a primitive shift on D.

2. Main results

The sets A and B are called *almost equal* if $(A \setminus B) \cup (B \setminus A)$ is finite. The sets A and B are called *almost disjoint* if $A \cap B$ is finite.

Let D be a countable set with discrete topology. A family $\mathcal{F}$ of infinite subsets of D is called *almost disjoint* family if every pair of elements of $\mathcal{F}$ is almost disjoint. The almost disjoint family $\mathcal{F}$ is called a *MAD* family or *maximal almost disjoint* family if it is maximal (in terms of inclusion) with respect to being almost disjoint.

Let $\mathcal{F}$ be an infinite almost disjoint family of infinite subsets of D. Let $X = D \cup \mathcal{F}$. We give a topology on X as follows: Each point of D is isolated in X. If $A \in \mathcal{F}$ then treat A also as a subset of D. A neighborhood base of A (treated as a point of X) is a collection of sets of the form $\{A\} \cup B$ where $B \subseteq A$ and $A \setminus B$ is finite. The space $D \cup \mathcal{F}$ is called *Ψ-like* if $\mathcal{F}$ is uncountable. A Ψ-like space is locally compact and not metrizable. Its one point compactification is called a *Ψ^*-like* space. If $\mathcal{F}$ is a MAD family then $D \cup \mathcal{F}$ is called a *Ψ-space* (see [3]) and its one point compactification is called a *Ψ^*-space*. We will use $\Psi(\mathcal{F})$ and $\Psi^*(\mathcal{F})$ respectively to denote these two spaces.

Let σ be a primitive shift on a countable discrete space D and let $\mathcal{F}$ be an almost disjoint family of infinite subsets of D. We say that $\mathcal{F}$ is *σ-invariant* if for every $A \in \mathcal{F}$ the sets $\sigma(A)$ and $\sigma^{-1}(A)$ are elements of $\mathcal{F}$.

Lemma 2.1. *Let σ be a primitive shift on a countable discrete space D. Then there is an infinite almost disjoint family of infinite subsets of D that is σ-invariant. Furthermore, any such family is contained in a maximal almost disjoint family that is σ-invariant.*

Proof. The primitive shift defines a well order on D by $a < b$ if $\sigma^n(b) = a$ for some positive integer n. Let A be an infinite subset of D, say $A = \{a_n\colon\ n = 1, 2, 3, \ldots\}$. Define set B as follows. We take a_1 as the first point in B. The next point is the least point a_k in A satisfying the conditions $a_1 < a_k$ and $a_1 = \sigma^j(a_k)$ for $j > 2$. We call this point a_{i_2} and the number j the distance between the two points.

Suppose that we have chosen the n-th point, a_{i_n}. We chose $a_{i_{n+1}}$ to be the least point in A satisfying $a_{i_n} < a_{i_{n+1}}$ and such that the distance between a_{i_n} and $a_{i_{n+1}}$ is greater than the distance between a_{i_n} and $a_{i_{n-1}}$. The family $\{\sigma^k(B)\colon\ k \text{ is an integer}\}$ is σ-invariant and almost disjoint.

Let $\mathcal{M}$ be the family of all (infinite) almost disjoint families of (infinite) subsets of D that are σ-invariant. Order $\mathcal{M}$ by inclusion. Let $\mathcal{G}$ be an element of $\mathcal{M}$, and let $\mathcal{C}$ be a maximal chain that contains $\mathcal{G}$. Let $\mathcal{F}$ be the union of $\mathcal{C}$. If $\mathcal{F}$ is not a maximal almost disjoint family, then there is an infinite subset A of D which is almost disjoint with every element of $\mathcal{F}$. Proceeding as above we can get an infinite subset B of A such that $\{\sigma^k(B)\colon\ k \text{ is an integer}\}$ is σ-invariant and almost disjoint. The union of the two families is greater than the family $\mathcal{F}$. So $\mathcal{F}$ is a maximal almost disjoint family. $\square$

Lemma 2.2. *Let D be a countable discrete space. There are $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of D.*

Proof. Let Q be the set of all rational numbers and let I be the set of all irrational numbers. First we show that there are $2^{\mathfrak{c}}$ maximal almost disjoint families on Q.

For every irrational number r let Q_r be the set of elements of a sequence of rational numbers converging to r. Note that the family $\{Q_r\colon\ r \in I\}$ is almost disjoint. Let $\mathcal{F}$ be a free ultrafilter on the set N of all positive integers of cardinality $2^{\aleph_0}$. So $\mathcal{F} = \{F_r\colon\ r \in I\}$.

Let $f_r : N \to Q_r$ be a one-to-one and onto function. Let σ be a permutation of I, that is $\sigma : I \to I$ is one-to-one and onto. Put $\mathcal{A}_\sigma = \{f_{\sigma(r)}(F_r)\colon\ r \in I\}$. Each $\mathcal{A}_\sigma$ is an almost disjoint family of infinite subsets of Q. If $\sigma_1 \neq \sigma_2$ then there are r_1 and r_2 such that $r_1 \neq r_2$ and $\sigma_1(r_1) = \sigma_2(r_2)$. So $f_{\sigma_1(r_1)}(F_{r_1})$ and $f_{\sigma_2(r_2)}(F_{r_2})$ are subsets of $Q_{\sigma_1(r_1)}$ and $f_{\sigma_1(r_1)}(F_{r_1}) \cap f_{\sigma_2(r_2)}(F_{r_2})$ is infinite. So $\mathcal{A}_{\sigma_1} \neq \mathcal{A}_{\sigma_2}$. There are $2^{\mathfrak{c}}$ permutations of irrational numbers, so there are $2^{\mathfrak{c}}$ maximal almost disjoint families of infinite subsets of Q. Therefore there are $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of D. $\square$

Theorem 2.3. *Let σ be a primitive shift on a countable discrete space D. There are $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of D that are σ-invariant.*

Proof. Let B be an infinite subset of D such that the family $\{\sigma^k(B)$: k is an integer$\}$ is σ-invariant and almost disjoint. Such a subset was constructed in the proof of Lemma 2.1.

Let $h : Q \to B$ be one-to-one and onto. In Lemma 2.2 we constructed $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of Q. Their images under h are maximal almost disjoint families of infinite subsets of B. Note that a union of any two of these images is no longer an almost disjoint family. Let $\mathcal{G}$ be any of these families. Then the family $\mathcal{G} = \{\sigma^n(C)$: $C \in \mathcal{G}$ and n is an integer$\}$ is an almost disjoint σ-invariant family. This family, by Lemma 2.1, can be extended to a maximal almost disjoint family of (infinite) subsets of D that is σ-invariant. This procedure can be applied to each of the images of $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of Q. So there are $2^{\mathfrak{c}}$ maximal almost disjoint families of (infinite) subsets of D that are σ-invariant. □

Corollary 2.4. *There are $2^{\mathfrak{c}}$ non-homeomorphic Ψ^*-spaces that admit a primitive shift.*

Proof. Let σ be a primitive shift on a countable discrete space D. Let $\mathcal{F}$ be a maximal almost disjoint family of (infinite) subsets of D that is σ-invariant. Let $\Psi(\mathcal{F})$ be a Ψ-space defined by $\mathcal{F}$. Then σ extends from D to $\Psi(\mathcal{F})$. There are $2^{\mathfrak{c}}$ maximal almost disjoint families of infinite subsets of D that are invariant under σ. So there are $2^{\mathfrak{c}}$ Ψ-spaces defined by these families. These spaces and their one point compactifications admit a primitive shift. Any homeomorphism of two of these spaces restricted to D is a permutation of elements of D. There are only $2^{\aleph_0}$ such permutations. So there are $2^{\mathfrak{c}}$ non-homeomorphic Ψ-spaces and Ψ^*-spaces that admit a primitive shift. □

The almost disjointness number $\mathfrak{a}$ is the minimal size of an (infinite) maximal almost disjoint family of (infinite) subsets of a countable set. The equality $\mathfrak{a} = \mathfrak{c}$ is consistent with ZFC (see [2]).

Theorem 2.5. *If $\mathfrak{a} = \mathfrak{c}$ then there exists a Ψ-space with no primitive shift.*

Proof. Let $\{\sigma_\alpha$: $\alpha < \mathfrak{c}\}$ be the family of all primitive shifts on a countable discrete space D. Let $\mathcal{A}$ be the family of all infinite subsets of D that have infinite complement. Any infinite almost disjoint family of infinite subsets of D is a subfamily of $\mathcal{A}$.

We will construct a sequence $\{\mathcal{F}_\alpha$: $\alpha < \mathfrak{c}\}$ of (infinite) almost disjoint families of infinite subsets of D such that

(1) $\mathcal{F}_\beta \subset \mathcal{F}_\alpha$ for $\beta < \alpha$;
(2) $\mathcal{F}_\alpha = \{A_\alpha, B_\alpha, C_\alpha\} \cup \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ for some pairwise almost disjoint sets $A_\alpha, B_\alpha, C_\alpha \in \mathcal{A}$ such that either $\sigma_\alpha(A_\alpha)$ has an infinite intersection with B_α and C_α or $\sigma_\alpha^{-1}(A_\alpha)$ has an infinite intersection with B_α and C_α.

Let $S_0 \in \mathcal{A}$. From the proof of Lemma 2.1 there is an infinite subset A_0 of S_0 such that $\{\sigma_0^k(A_0)$: k is an integer$\}$ is an almost disjoint family. Put B_0 and C_0 to be two infinite disjoint subsets of $\sigma_0(A_0) \setminus A_0$. Let $\mathcal{F}_0 = \{A_0, B_0, C_0\} \cup \{\sigma_0^k(A_0)$: k is an integer and $k \neq 1\}$. So $\mathcal{F}_0$ is an infinite almost disjoint family and conditions (1) and (2) are satisfied.

Suppose we have constructed almost disjoint families $\mathcal{F}_\beta$, for $\beta < \alpha$, that satisfy conditions (1) and (2). Note that $\mathcal{F}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ is an infinite almost disjoint family and its cardinality is $\omega \cdot \alpha < \mathfrak{c} = \mathfrak{a}$, so it is not maximal. So there is S_α in $\mathcal{A}$ that is almost disjoint with every element of $\mathcal{F}_{<\alpha}$. If $\sigma_\alpha(S_\alpha)$ is almost disjoint with every element of $\mathcal{F}_{<\alpha}$ then we construct the three sets as before, that is let A_α be an infinite subset of S_α such that $\{\sigma_\alpha^k(A_\alpha)$: k is an integer$\}$ is an almost disjoint family. Put B_α and C_α to be two infinite disjoint subsets of $\sigma_\alpha(A_\alpha)$. Put $\mathcal{F}_\alpha = \mathcal{F}_{<\alpha} \cup \{A_\alpha, B_\alpha, C_\alpha\}$. Then conditions (1) and (2) are satisfied.

If for every S_α in $\mathcal{A}$ that is almost disjoint with every element of $\mathcal{F}_{<\alpha}$ we have that $\sigma_\alpha(S_\alpha)$ has an infinite intersection with some element of $\mathcal{F}_{<\alpha}$ we take A_α to be that element. So $\sigma_\alpha^{-1}(A_\alpha) \cap S_\alpha$ is an infinite subset of S_α. We split it into two infinite disjoint subsets that we call B_α and C_α. Put $\mathcal{F}_\alpha = \mathcal{F}_{<\alpha} \cup \{A_\alpha, B_\alpha, C_\alpha\}$. Then conditions (1) and (2) are satisfied.

The family $\bigcup_{\beta<\mathfrak{c}} \mathcal{F}_\beta$ is almost disjoint and is not invariant with respect to any primitive shift on D. There is a maximal almost disjoint family $\mathcal{F}$ that contains it.

We claim that spaces $\Psi(\mathcal{F})$ and $\Psi^*(\mathcal{F})$ admit no primitive shift. Suppose there is a primitive shift $\sigma : \Psi(\mathcal{F}) \to \Psi(\mathcal{F}) \setminus \{p\}$. The restriction $\sigma|_D$ is a primitive shift on D, and therefore $\sigma = \sigma_\alpha$ for some $\alpha < \mathfrak{c}$. Consider the sets $\{A_\alpha, B_\alpha, C_\alpha\}$ in $\mathcal{F}_\alpha$. These sets are simultaneously points of the family $\mathcal{F} \subset \Psi(\mathcal{F})$. There are two cases to consider.

First suppose that B_α and C_α are subsets of $\sigma_\alpha(A_\alpha)$. In this case we consider the image $A = \sigma(A_\alpha)$ in $\Psi(\mathcal{F}) = D \cup \mathcal{F}$ of the point $A_\alpha \in \mathcal{F}$. So A is an element of $\mathcal{F}$. Since B_α and C_α are distinct elements in $\mathcal{F}$ the point A differs from either B_α or C_α. Assume that $A \neq B_\alpha$. Then $A \cap B_\alpha$ is finite.

The set $\mathcal{O}(A) = \{A\} \cup (A \setminus B_\alpha)$ is a neighborhood of the point A in $\Psi(\mathcal{F})$. By the continuity of σ at the point A_α there is a neighborhood $\mathcal{O}(A_\alpha)$ of A_α such that $\sigma(\mathcal{O}(A_\alpha)) \subset \mathcal{O}(A)$. By the definition of the topology on $\Psi(\mathcal{F})$ the set $A_\alpha \setminus \mathcal{O}(A_\alpha)$ is finite. Since $\sigma_\alpha^{-1}(B_\alpha) \subset A_\alpha$ is infinite, there is an element b in $\sigma_\alpha^{-1}(B_\alpha) \cap \mathcal{O}(A_\alpha)$. For this point we get $\sigma(b) \in B_\alpha \cap \mathcal{O}(A)$ which is not possible as $B_\alpha \cap \mathcal{O}(A) = \emptyset$.

If B_α and C_α are subsets of $\sigma_\alpha^{-1}(A_\alpha)$ we proceed as above, replacing σ by σ^{-1}.

So $\Psi\{\mathcal{F}\}$ is a Ψ-space with no primitive shift. □

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